BULLETIN

OF THE

AMERICAN MATHEMATICAL SOCIETY

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VOLUME XXXVIII, NUMBER 12

DECEMBER, 1932

PUBLISHED BY THE SOCIETY
MENASHA, WIS., AND NEW YORK
1932

PUBLISHED MONTHLY

Whole No. 399 $9.00 a Year

Entered as second class matter April 24, 1926 at the post office at Menasha, Wis., under the act of August 24, 1912.

AMERICAN MATHEMATICAL SOCIETY

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CESÀRO SUMMABILITY OF DOUBLE SERIES

BY MARGARET GURNEY

1. *Definitions and Notation.* The familiar Cesàro transform of a double series $\sum_{i,j=1}^{\infty} u_{ij}$ is given by

$$\tilde{S}_{mn}^{(\alpha,\beta)} = S_{mn}^{(\alpha,\beta)} \Big/ \left[\binom{m+\alpha-1}{\alpha}\binom{n+\beta-1}{\beta}\right], \tag{1}$$

where

$$S_{mn}^{(\alpha,\beta)} = \sum_{i,j=1}^{m,n} \binom{m+\alpha-i}{\alpha}\binom{n+\beta-j}{\beta} u_{ij}. \tag{2}$$

The series $\sum u_{ij}$ is said to be summable $(C;\alpha,\beta)$ to S if we have $\lim_{m,n\to\infty} \tilde{S}_{mn}^{(\alpha,\beta)} = S$; to be bounded $(C;\alpha,\beta)$ if $|\tilde{S}_{mn}^{(\alpha,\beta)}| < \text{const.}$ for all values of m and n; and to be ultimately bounded $(C;\alpha,\beta)$ if $\limsup_{m,n\to\infty} |\tilde{S}_{mn}^{(\alpha,\beta)}| < \infty$. This definition holds for all values of α and β, real or complex, except negative integers, the binomial coefficients being defined as usual in terms of the gamma function; however we shall be concerned only with real orders greater than -1.

A special but important type of double series is that for which u_{ij} is factorable, say

$$u_{ij} = v_i w_j, \qquad (i, j = 1, 2, 3, \cdots). \tag{3}$$

Defining

$$V_m^{(\alpha)} = \sum_{i=1}^{m} \binom{m+\alpha-i}{\alpha} v_i; \quad W_n^{(\beta)} = \sum_{j=1}^{n} \binom{n+\beta-j}{\beta} w_j, \tag{4}$$

we have, by (2), when (3) holds, $S_{mn}^{(\alpha,\beta)} = V_m^{(\alpha)} W_n^{(\beta)}$, so that, by (1),

$$\tilde{S}_{mn}^{(\alpha,\beta)} = \tilde{V}_m^{(\alpha)} \tilde{W}_n^{(\beta)}, \tag{5}$$

where the factors in the right member of (5) are, respectively, the (C,α) transform of $\sum v_i$ and the (C,β) transform of $\sum w_j$.

2. *Examples.* The relation (5) enables us to obtain very easily examples illustrating the following statements.

THEOREM 1. *There is a series* $\sum u_{ij}$ *which is summable and bounded* $(C; \alpha,\beta)$ *for every* $\alpha>0$, $\beta>0$, *while* (a) *each row and column of* $\sum u_{ij}$ *has bounded partial sums and* (b) *each row and column of* $\sum u_{ij}$ *is non-summable* (C, γ) *for every* $\gamma>-1$.

Let $v_1=-1$ and $v_i=2(-1)^i$ when $i>1$; then $\sum_{i=1}^{m} v_i=(-1)^m$ and $\sum v_i$ is, as is well known, summable (C, δ) to 0 for every $\delta>0$.* Let $\sum w_j$ be any series whose partial sums are bounded, and which is non-summable (C, γ) for every $\gamma>-1$.† It is easy to show that the series whose general term is given by

$$u_{ij} = v_i w_j + w_i v_j$$

is summable $(C;\alpha,\beta)$ to 0, and satisfies the other conditions of Theorem 1.

THEOREM 2. *Corresponding to each pair of numbers* α *and* β, $\alpha>-1$, $\beta>-1$, *there is a series* $\sum u_{ij}$ *which is summable and bounded* $(C; \alpha,\beta)$ *while* (a) *each row* [*column*] *is unbounded* $(C, \beta-\delta)$ $[(C, \alpha-\delta)]$, (b) *each row* [*column*] *is non-summable but bounded* (C, β) $[(C, \alpha)]$ *and* (c) *each row* [*column*] *is summable* $(C, \beta+\delta)$ $[(C, \alpha+\delta)]$, *for every* $\delta>0$.

Let $v_1=-1$ and $v_i=2(-1)^i$ when $i>1$, as before. Corresponding to a number $\gamma>-1$, let $\sum w_i^{(\gamma)}$ be the series having

$$\sum_{i=1}^{p}(-1)^i\binom{i+\gamma-1}{\gamma}, \quad (p = 1, 2, 3, \cdots),$$

for its sequence of partial sums. Then $\sum w_i^{(\gamma)}$ has‡ an unbounded $(C, \gamma-\delta)$ transform, a non-convergent but bounded (C, γ) transform, and a convergent $(C, \gamma+\delta)$ transform. These facts and properties of $\sum v_i$ enable us to show that the series whose general term is given by $u_{ij}=v_i w_j^{(\beta)}+w_i^{(\alpha)} v_j$ is summable $(C;\alpha,\beta)$ to 0 and fulfills the other conditions of Theorem 2.

* Σv_i is bounded $(C, 0)$ and summable $(C, 1)$ to 0; it is therefore summable (C, δ) to 0 for every $\delta>0$. See Zygmund, *Sur un théorème de la théorie de la sommabilité*, Mathematische Zeitschrift, vol. 25 (1926), p. 291.

† An example of such a series is Σw_j, where w_j is the coefficient of x^j in the series $\sum_{r=0}^{\infty}(-1)^r x^{r!}$. See Hardy, *On certain oscillating series*, Quarterly Journal of Mathematics, vol. 38 (1906–7), p. 286. Hardy's result shows that $\sum w_j$ is not Abel summable, and non-summability (C, γ) for every $\gamma>-1$ follows.

‡ See Knopp, *Unendliche Reihen*, 3d edition, 1931, p. 496, Ex. 2. The discussion is given for $\gamma=k$, an integer, but it holds also for any real $\gamma>-1$.

3. *Necessary Conditions for Summability.* We now give two necessary conditions for summability of double series.

THEOREM 3. *If a double series is ultimately bounded* $(C; k, l)$, *k and l being fixed positive integers, then each sufficiently advanced row* [*column*] *is bounded* (C, l) $[(C, k)]$.

By the ultimate boundedness $(C; k, l)$ there exist constants K, M, and N such that $|\bar{S}^{(k,l)}_{mn}| < K$ for $m > M$, $n > N$. Fix $m > M+k+1$; then for every $n > N$, $|\bar{S}^{(k,l)}_{m-r,n}| < K$, for $r = 0, 1, 2, \cdots, k+1$. Moreover, we have

$$|S^{(k,l)}_{m-r,n}| \Big/ \binom{m+k-1}{k}\binom{n+l-1}{l} = |\bar{S}^{(k,l)}_{m-r,n}| \binom{m+k-r-1}{k} \Big/ \binom{m+n-1}{k} < K,$$

and

$$\left| \sum_{r=0}^{k+1} (-1)^r \binom{k+1}{r} S^{(k,l)}_{m-r,n} \Big/ \binom{m+k-1}{k}\binom{n+l-1}{l} \right| < K \sum_{r=0}^{k+1} \binom{k+1}{r} = 2^{k+1}K.$$

The product of $\binom{m+k-1}{k}$ by the sum on the left is the (C, l) transform of the mth row of the double series.* This row is a simple series which is ultimately bounded (C, l) and therefore is bounded (C, l) as was to be shown.

THEOREM 4. *If a double series is summable* $(C; k, l)$, *k and l being fixed positive integers, then each sufficiently advanced row* [*column*] *is either* (a) *summable* $(C, l+\delta)$ $[(C, k+\delta)]$ *for every* $\delta > 0$ *or* (b) *non-summable* (C, γ) *for every* $\gamma > -1$.

This follows from Theorem 3 and the result of Zygmund, loc. cit.

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* This is seen by equating coefficients of $x^m y^n$ in the identity
$$(1-x)^{k+1} \sum_{i,j=1}^{\infty} S^{(k,l)}_{ij} x^i y^j = (1-y)^{-(l+1)} \sum_{i,j=1}^{\infty} u_{ij} x^i y^j.$$

RESULTS AND PROBLEMS ABOUT n-WEBS OF CURVES IN A PLANE

BY WILHELM BLASCHKE

If

$$x^* = u(x, y), \qquad y^* = v(x, y)$$

is a topological mapping of the x, y plane, we call the function u *topologically equivalent* to x in this plane. Let us assume n such functions $t_i(x, y)$, $(i=1, 2, \cdots, n)$, in the same simply connected domain D. We have there n *sheaves* of curves $t_i(x, y)=\text{const}$. We call this figure an *n-web* if two curves of different sheaves have not more than one point in common. We suppose the functions $u_{ik}(t_i)$ to be continuous and *strictly monotonic*, so that for all pairs $t_i \neq t_i'$, we have $u_{ik}(t) \neq u_{ik}(t')$.

It seems to be interesting to study n-webs satisfying the condition that there are such functions $u_{ik}(t_i)$ satisfying identically in D the relations

$$\sum_{i=1}^{n} u_{ik}(t_i) = \text{const.}, \quad (k = 1, 2, 3, \cdots, m). \tag{1}$$

We call these equations (1) linearly independent, if the identities

$$\sum_{k=1}^{m} c_k u_{ik}(t_i) = \text{const.} \tag{2}$$

imply for the constants c_k the trivial solution $c_k=0$, $(k=1, 2, 3, \cdots, m)$. The following theorems hold.

THEOREM 1. *A 3-web satisfying one condition* (1), $(n=3, m=1)$, *is topologically equivalent to the tangents of a curve of class* 3 (*irreducible or not*).

This was essentially found by Graf and Sauer† in 1924. Howe and I‡, in 1932, proved the following theorem.

† H. Graf and R. Sauer, Münchener Berichte, 1924; W. Blaschke and G. Howe, Hamburg Abhandlungen, vol. 9 (1932); W. Blaschke, Tôhoku Mathematical Journal, 1933.

‡ W. Blaschke and G. Howe, Hamburg Abhandlungen, vol. 9 (1932).

THEOREM 2. *A straight lined n-web satisfying (at least) one condition* (1) *is necessarily equivalent to the tangents of a curve of class n* ($n \geqq 3$).

Our result contains Theorem 1 as a special case, because a 3-web satisfying the equation

$$u_1 + u_2 + u_3 = \text{const.}$$

is equivalent to a special straight lined 3-web (*hexagonal web*), as we see if we assume u_1 and u_2 as parallel coordinates.

Howe observed that the following Theorem 3 is equivalent to S. Lie's results about the surfaces, which are *translation surfaces* in different ways.

THEOREM 3. *A 4-web satisfying 3 linearly independent relations* ($n=4$, $m=3$) *is equivalent to the tangents of a curve of class* 4.

A geometric interpretation of one condition (1) for a 4-web has been given by Bose and myself.† Bol‡ discovered a short time ago the following result.

THEOREM 4. *The maximum number m of linearly independent relations* (1) *for an n-web is*

$$m = \frac{(n-1)(n-2)}{2}. \tag{3}$$

Almost equivalent to a theorem of Reidemeister§ are the following.

THEOREM 5. *A 4-web satisfying 3 linearly independent relations* (1) *with* $u_{ii}=0$ *is equivalent to 4 pencils of straight lines, no 3 of the 4 vertices on a straight line.*

THEOREM 6. *A 4-web satisfying 3 relations* (1) *with* $u_{ii}=0$, ($i=1, 2, 3$), *only two of them linearly independent, admits a continuous one-parameter group, the* $t_4=\text{const.}$ *being paths.*

But between the proofs of these theorems there is an essential difference. Only Theorem 1 and the greater part of Theorems 5 and 6 are proved without any further restrictions for the functions t_i, u_{ik}. The proofs already known for Theorems 2–6

† W. Blaschke and R. C. Bose, Indian Physico-Mathematical Journal, vol 3 (1932), p. 99.

‡ G. Bol, this Bulletin, vol. 38 (1932), pp. 855–857.

§ K. Reidemeister, Mathematische Zeitschrift, vol. 29 (1928).

contain regularity restrictions. Therefore the first problem to be solved is the following one.

PROBLEM A. *Do the Theorems* 2–5 *remain valid without further regularity restrictions?*

Another question unsolved as far as I know is the following one.

PROBLEM B. *To extend our Theorem* 3 *to n-webs.*

These problems seem to be interesting because, for example, they contain a kind of real geometrical interpretation of Abel's theorem on algebraic curves.

Finally a few words about more dimensions. The questions about *webs of surfaces*

$$S_i(x, y, z) = \text{const.}$$

in a 3-space can partially be reduced to our theorems on curve-webs in a 2-space. But if we consider *sheaves of curves*

$$s_i(x, y, z) = \text{const.}, \; t_i(x, y, z) = \text{const.}$$

in a 3-space we may ask, for example, the following question.

PROBLEM C. *How many essentially different relations*

$$u_1(s_1, t_1) + u_2(s_2, t_2) + u_3(s_3, t_3) = \text{const.}$$

can exist for a 3*-web of curves*

$$s_i, t_i = \text{const.}, \qquad (i = 1, 2, 3),$$

in a 3*-space?*

This seems to me to be one of the most promising fields of geometric research.

THE UNIVERSITY OF CHICAGO, AND
THE UNIVERSITY OF HAMBURG

SOLUTION OF THE ZARANKIEWICZ PROBLEM*

BY E. W. MILLER

1. *Introduction.* In 1925 C. Zarankiewicz† proposed the following problem: *Is every acyclic continuous curve‡ homeomorphic with some proper subset of itself?* It is the purpose of this paper to show that the above question is to be answered in the negative.

Our result will depend upon the following theorem.

THEOREM. *The acyclic continuous curve S is homeomorphic with no proper subset of itself if it contains a set K such that* (1) *each point of K is a fixed point with respect to any* (1, 1) *bicontinuous transformation of S into a subset of itself; and* (2) *each point of S of (Urysohn-Menger) order >1 lies on an arc of S whose end points are points of K.*

PROOF. Let p be any point of S of order >1. There is an arc a_1a_2 in S which contains p and whose end points are points of K. Let T be any (1, 1) bicontinuous transformation of S into a subset of itself. Since $T(a_1)=a_1$ and $T(a_2)=a_2$, and since there is just one arc in S from a_1 to a_2, T must carry a_1a_2 into itself. Hence there is a point q of a_1a_2 such that $T(q)=p$. Thus the subset of S into which T carries S must contain all points of S of order >1. As these points are dense in S, this subset must be S itself.

Our problem, then, is to construct an acyclic continuous curve which satisfies the conditions of the above theorem. We shall first define certain auxiliary sets $E_{x_1x_2\cdots x_k}$.

2. *Definition of the Sets $E_{x_1x_2\cdots x_k}$.* Within a linear interval ab choose points a_n so that $a_{n+1}<a_n$ and $\lim a_n=a$. Within each interval $a_{n+1}a_n$ choose points $a_{n,m}$ so that $a_{n,m}<a_{n,m+1}$ and $\lim_{m=\infty} a_{n,m}=a_n$. At each point a_n and $a_{n,m}$ erect a perpendicular to ab. Take these perpendiculars so that for any $\epsilon>0$ only a finite number of them have a length $>\epsilon$. The set of points obtained in this way will be called a set E_1. The point a will be

* Presented to the Society, October 29, 1932.

† Fundamenta Mathematicae, vol. 7, p. 381, problem 37.

‡ The term *continuous curve* is used throughout the present article to mean a compact, locally connected, metric continuum.

called the *origin* of E_1 and the perpendiculars which we have erected will all be referred to as *perpendiculars of rank* 1. It is clear that E_1 is an acyclic continuous curve. Everything will be the same in the definition of E_2 except for this one change: the points $a_{n,m}$ are taken within $a_{n+1}a_n$ so that $a_{n,m+1}<a_{n,m}$ and $\lim_{m=\infty} a_{n,m}=a_{n+1}$.

Let us now suppose that we have defined sets $E_{x_1x_2\cdots x_i}$, where x_i $(i\leqq k)$ can have either of the two values 1 and 2. Let us suppose furthermore that we have defined the expressions *origin of* $E_{x_1x_2\cdots x_k}$ and *perpendiculars of rank k of* $E_{x_1x_2\cdots x_k}$. We suppose finally that $E_{x_1x_2\cdots x_k}$ has been so defined that it is an acyclic continuous curve. To define the set $E_{x_1x_2\cdots x_k1}$, we proceed as follows. We replace each perpendicular of rank k of $E_{x_1x_2\cdots x_k}$ by a set E_1 whose origin is the foot of that perpendicular. Furthermore we do this, as we clearly can, so that the resulting set $E_{x_1x_2\cdots x_k1}$ is an acyclic continuous curve. By *origin of* $E_{x_1x_2\cdots x_k1}$ we will mean merely the origin of $E_{x_1x_2\cdots x_k}$, and by *perpendiculars of rank* $k+1$ *of* $E_{x_1x_2\cdots x_k1}$ the perpendiculars of rank 1 of the sets E_1 employed in obtaining $E_{x_1x_2\cdots x_k1}$ from $E_{x_1x_2\cdots x_k}$. Everything will be the same in the definition of $E_{x_1x_2\cdots x_k2}$ except for this one change: in obtaining $E_{x_1x_2\cdots x_k2}$ from $E_{x_1x_2\cdots x_k}$ we shall employ sets E_2 instead of sets E_1.

3. *Construction of an Acyclic Continuous Curve which Satisfies the Conditions of the Theorem.* This construction will be achieved through the use of the following sequence of sets:

$$E_1, E_{21}, E_{221}, \cdots, E_{22\cdots21}, \cdots.$$

Let us first re-label these sets in the order named as

$$W_1, W_2, W_3, \cdots, W_n, \cdots.$$

We begin with a set W_1 whose origin is a point a and adjoin to it three line segments ac, ad, and ae so that the only point which any two of the sets W_1, ac, ad, and ae have in common is the point a. Let us denote the resulting acyclic continuous curve by S_1. We now consider the arc in S_1 from each end point of S_1 to a. There will be a first branch point of S_1 in the order from the end point of S_1 to a on such an arc, and the portion of the arc from the end point of S_1 to this branch point is a line segment. Denote the mid-point of this segment by x. We obtain, of course,

a countable infinity of points x. With this countable infinity of points we associate in a (1, 1) way* the sets of odd index

$$W_3, W_5, W_7, \cdots$$

and take x as the origin of the associated set $W(x)$ in such a way that S_1 and $W(x)$ have only the point x in common. Also we attach to the point x a straight line segment having x as one end point and having only x in common with $W(x)+S_1$. All this can clearly be done so that the resulting set S_2 is an acyclic continuous curve. Now S_3 will be related to S_2 in the same way as S_2 is related to S_1, except that we make use of sets $W_{2(2m+1)}$ instead of sets W_{2m+1}. In general S_{n+1} is related to S_n in the same way as S_n is related to S_{n-1}, except that we make use of sets $W_{2^{n-1}(2m+1)}$ instead of sets $W_{2^{n-2}(2m+1)}$. Now constructions of the general type just described are common in the literature and it is well known† that such a construction can be carried through so that the closure of the sum of the acyclic continuous curves successively obtained is itself an acyclic continuous curve. We may suppose then that $S=(\overline{\sum_{n=1}^{\infty} S_n})$ is an acyclic continuous curve.

It will now be shown that S satisfies the conditions of our theorem. We notice first that any branch point of S is either of order 3 or order 4. The points of order 4 are the point a of S_1 and the points x which arise at successive stages of our process of construction. We will denote the set of points of order 4 of S by K, and it will be shown that K has the properties of the set K in our theorem. In fact, it is obvious from the way in which S was constructed that K has property (2) of the theorem. We need only show that it has property (1).

In the first place, we notice that if T is any (1, 1) bicontinuous transformation of S into a subset of itself, T must carry each point of K into a point of K, since no point of S is of order >4 and K contains all points of order 4 of S. Let us suppose that there are two *distinct* points q_1 and q_2 of K such that $T(q_1)=q_2$. Let us suppose for definiteness (the argument is similar in the opposite case) that the set W which has q_1 as its origin is of

* It is clear that this (1, 1) correspondence can be made perfectly definite.

† For a similar construction and proof that the result is an acyclic continuous curve see K. Menger, Fundamenta Mathematicae, vol. 10 (1927), p. 108.

lower index than the set W which has q_2 as its origin. If we consider any point q of K we notice that of the four essentially distinct arcs of S which meet in q just one has the property that the branch points on it have q as limit point. Let us denote this arc by qb and take the point b so close to q that qb is a line segment and contains no point of order 4 other than q. Let us now consider the arcs q_1b_1 and q_2b_2. It is clear that there is a sub-arc q_1b_1' of q_1b_1 and a sub-arc q_2b_2' of q_2b_2 such that the transform of q_1b_1' is q_2b_2'. Any branch point of S on q_1b_1' is transformed into a branch point of S on q_2b_2'. If $W(q_1) = W_1$, we see that we have already reached a contradiction. For $W_1 = E_1$ and $W(q_2) = W_m = E_{2\ldots1}$, which means that q_1b_1' contains branch points which are limit points of branch points from *the left*, while q_2b_2' contains no such points. If $W(q_1) = W_2$, we fix our attention upon some one branch point of S interior to q_1b_1'. Let us denote this point by r_1 and the corresponding point on q_2b_2' by r_2, and consider the perpendiculars to q_1b_1' and q_2b_2' at r_1 and r_2, respectively. Denote these perpendiculars by r_1s_1 and r_2s_2. Now since $W_2 = E_{21}$, r_1 is a limit point along r_1s_1 of branch points of S which are in turn limit points of branch points of S from *below* along r_1s_1, while r_2s_2 contains no such points since $W(q_2) = W_m = E_{22\ldots1}$. It is obvious that the argument exemplified above can be extended to apply to the general case where $W(q_1) = W_n$ and $W(q_2) = W_m$ whether $n < m$ or $m < n$. It follows that $q_1 = q_2$ if q_1 and q_2 belong to K and $T(q_1) = q_2$.

In conclusion it may be remarked that it is possible to construct an acyclic continuous curve which contains no point of order >3 and which is homeomorphic with no proper subset of itself. We need only employ sets E_{212}, E_{2112}, E_{21112}, $\cdots$ instead of sets E_1, E_{21}, E_{221}, $\cdots$. The proof will involve only a few more details than the proof given here.

THE UNIVERSITY OF MICHIGAN

A PROPERTY RELATED TO COMPLETENESS*

BY J. H. ROBERTS

In 1926, R. L. Moore presented the following axiom.

AXIOM 1′. *There exists a countable sequence* $G_1, G_2, G_3, \cdots$ *such that* (a) *for each* n, G_n *is a collection of domains covering space,* (b) *if* P_1 *and* P_2 *are distinct points of a domain* R, *there exists an integer* d *such that if* $n>d$ *and* K_n *is a domain containing* P_1 *and belonging to* G_n, *then* $\overline{K}_n$ *is a subset of* $R-P_2$, *and* (c) *if* $R_1, R_2, R_3, \cdots$ *is a sequence of domains such that, for each* n, R_n *belongs to* G_n *and such that, for each* n, $R_1, R_2, \cdots, R_n$ *have a point in common, then there exists a point common to all the point sets* $\overline{R}_1, \overline{R}_2, \overline{R}_3, \cdots$.†

Moore has given an example of a non-metric space in which his axiom 1′ holds true. He raised the question as to whether or not a metric space in which his axiom 1′ holds true is *complete.*‡ The present paper answers this question in the affirmative.§

THEOREM. *A metric space* S *in which axiom* 1′ *holds true is complete.*

PROOF. Let $\delta(x, y)$ be a distance function defined over the space S. Let P be any point of S and let n be any positive integer. Either (1) there is a domain of the set G_n which contains every point y such that $\delta(P, y)\leqq 2$, or (2) there exists a greatest number k ($k\leqq 2$) such that if $r<k$, then there exists a domain of the set G_n containing every point y such that $\delta(P, y)\leqq r$. Let

* Presented to the Society, April 11, 1931.

† This Bulletin, vol. 33 (1927), pp. 141.

‡ A space S is said to be *complete* if there exists a definition of distance such that every sequence of points satisfying the Cauchy condition has a limit point. A sequence of points $P_1, P_2, \cdots$, in a metric space is said to satisfy the Cauchy condition with respect to the distance function δ if, for every positive number e, there exists an integer n such that $\delta(P_n, P_k)<e$ if $k>n$.

§ The present result was obtained about September 1, 1930, and was reported to Professor Moore at that time. I purposely delayed publishing the paper in order that it might not appear in advance of the publication of his book *Foundations of Point Set Theory*. Later in the fall of 1930 Leo Zippin obtained a theorem which, with *other theorems in the literature,* yields the result of this paper.

$2w$ denote 2 or k according as the first or second condition holds true. Let $C(P, n)$ denote the set of all points y such that $\delta(P, y) \leqq w$. The number w will be called the *radius of* $C(P, n)$.

NOTATION. If x, y, and Q are points, let $f(x, y; Q)$ denote the maximum of the two quantities $\delta(x, Q)$ and $\delta(y, Q)$. If n is a positive integer, let $r(Q, n)$ denote the radius of $C(Q, n)$.

Let x and y be any two points. We shall define a function $d_n(x, y)$. If there exists no point P such that $C(P, n)$ contains both x and y, then $d_n(x, y) = 1$. If there is a point P such that $C(P, n)$ contains both x and y, then let $e_{Pn}(x, y)$ be defined as $[f(x, y; P)/r(P, n)]^{1/3}$ and let $d_n(x, y)$ be the minimum or greatest lower bound of the numbers $e_{Pn}(x, y)$, where P can be any point such that $C(P, n)$ contains both x and y. A function $\rho(x, y)$ is now defined as follows:

$$\rho(x, y) = \delta(x, y) + \sum_{n=1}^{\infty} d_n(x, y)/2^n. \tag{1}$$

It is to be shown that $\rho(x, y)$ is a distance function with respect to which S is complete. Clearly $\rho(x, y) = \rho(y, x)$ and $\rho(x, x) = 0$. If $\rho(x, y) = 0$, then $\delta(x, y) = 0$ and $x = y$. Suppose that the point x is a limit point of the point set M. Let ϵ be any positive number. Then there exists a positive integer n such that $1/2^n < \epsilon/3$. There exists a point y of M such that $\delta(x, y) < r(x, i)$ and that $[\delta(x, y)/r(x, i)]^{1/3} < \epsilon/3$ for every integer i, $(i \leqq n)$. Then since $d_i(x, y) \leqq [\delta(x, y)/r(x, i)]^{1/3}$ it follows that $d_i(x, y) < \epsilon/3$, $(i \leqq n)$. Hence

$$\begin{aligned} \rho(x, y) &\leqq \delta(x, y) + \sum_{i=1}^{n} d_i(x, y)/2^i + \sum_{i=n+1}^{\infty} 1/2^i \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

Thus if x is a limit point of M, then for every positive number ϵ there is a point y of M such that $\rho(x, y) < \epsilon$. If x is not a limit point of M, then $\rho(x, M) \geqq \delta(x, M) > 0$, and there exists a positive number ϵ such that, if y is any point of M, then $\rho(x, y) > \epsilon$.

Let x, y, and z be any three points of S. It is to be shown that $\rho(x, y) + \rho(y, z) \geqq \rho(x, z)$. It is sufficient to show that for every n, $d_n(x, y) + d_n(y, z) \geqq d_n(x, z)$. Suppose that for some n we have $d_n(x, y) + d_n(y, z) < d_n(x, z)$, where x, y and z are distinct points. From the definition of the function d_n it follows that there exist

points P and Q such that $C(P, n)$ contains x and y, $C(Q, n)$ contains y and z, and

$$(2) \qquad \left[\frac{f(x, y; P)}{r(P, n)}\right]^{1/3} + \left[\frac{f(y, z; Q)}{r(Q, n)}\right]^{1/3} < d_n(x, z).$$

Let A denote $d_n(x, z)$. Let t and r denote, respectively, the first and second terms of the left member of (2). We next prove that

$$(3) \qquad A^3 \leqq \left[\frac{\delta(z, P)}{r(P, n)}\right].$$

This is obvious if z is not in $C(P, n)$ [that is, $\delta(z, P) \geqq r(P, n)$] since $A \leqq 1$. Let us suppose that z is in $C(P, n)$, and also that $\delta(z, P) \leqq \delta(x, P)$. Then $d_n(x, z) \leqq [\delta(x, P)/r(P, n)]^{1/3}$, which contradicts (2). Hence $\delta(z, P) > \delta(x, P)$. Then $[\delta(z, P)/r(P, n)]^{1/3}$ is one of the quantities of which $d_n(x, z)$, that is, A, is a lower bound. Hence in any case (3) is established. Similarly

$$(4) \qquad A^3 \leqq \frac{\delta(x, Q)}{r(Q, n)}.$$

The following two inequalities obviously hold, since δ is a distance function:

$$(5) \qquad \delta(z, P) \leqq \delta(z, Q) + \delta(Q, y) + \delta(y, P),$$

$$(6) \qquad \delta(x, Q) \leqq \delta(x, P) + \delta(P, y) + \delta(y, Q).$$

From the definition of t and of r, and the fact that $r < A - t$, the following hold true, where $v_1 = r(P, n)$ and $v_2 = r(Q, n)$:

$$(7) \qquad \begin{gathered} t^3 \geqq \frac{\delta(x, P)}{v_1}, \qquad t^3 \geqq \frac{\delta(y, P)}{v_1}, \\ (A - t)^3 \geqq \frac{\delta(y, Q)}{v_2}, \quad (A - t)^3 \geqq \frac{\delta(z, Q)}{v_2}. \end{gathered}$$

If now (5) is divided by v_1 and substitutions are made from (3) and (7) we have

$$(8) \qquad A^3 \leqq \frac{2(A - t)^3 v_2}{v_1} + t^3.$$

Likewise if (6) is divided by v_2 we have after substituting

$$A^3 \leqq \frac{2t^3 v_1}{v_2} + (A - t)^3 \cdot \tag{9}$$

Now (8) and (9) cannot both hold under the conditions here obtaining, namely $1 \geqq A > t > 0$, $v_1 > 0$, $v_2 > 0$. For from (9) we get $v_1/v_2 \geqq [A^3 - (A-t)^3]/2t^3$, from which, by (8), we find

$$3A^4 - 3A^2t^2 + 6At^3 - 3t^4 \leqq 0. \tag{10}$$

Now set $A = kt$. Then $k > 1$, and we have, after dividing by t^4, $3k^4 - 3k^2 + 6k - 3 \leqq 0$. This is obviously false. Hence our supposition has led to a contradiction, whence it follows that for every n, $d_n(x, y) + d_n(y, z) \geqq d_n(x, z)$. We can now say that the function $\rho(x, y)$ is a *distance* function.

The problem remains to show that with respect to this definition of distance the space S is complete. Let $P_1, P_2, P_3, \cdots$ denote any sequence of points satisfying the Cauchy condition with respect to the distance function $\rho(x, y)$. Let n be any positive integer and let m_n be an integer such that $\rho(P_k, P_h) < 1/2^{n+2}$ if $h, k \geqq m_n$. Then $d_n(P_h, P_k)/2^n < 1/2^{n+2}$, whence $d_n(P_h, P_k) < 1/4$ if $h, k \geqq m_n$. There exists an integer a $(a \geqq m_n)$ such that for every b, $(b \geqq m_n)$, $\delta(P_{m_n}, P_b) < 2\delta(P_{m_n}, P_a)$. Now we have $d_n(P_{m_n}, P_a) < 1/4$. Hence there exists a point Q such that

$$\left[\frac{\delta(P_{m_n}, Q)}{r(Q, n)}\right]^{1/3} < \frac{1}{4} \quad \text{and} \quad \left[\frac{\delta(P_a, Q)}{r(Q, n)}\right]^{1/3} < \frac{1}{4} \cdot$$

Then $\delta(P_{m_n}, Q) < r(Q, n)/64$, $\delta(P_a, Q) < r(Q, n)/64$, and thus $\delta(P_{m_n}, P_a) < r(Q, n)/32$. Then

$$\delta(P_b, Q) \leqq \delta(P_b, P_{m_n}) + \delta(P_{m_n}, Q) \leqq 2\delta(P_{m_n}, P_a) + \delta(P_{m_n}, Q)$$
$$< \frac{r(Q, n)}{16} + \frac{r(Q, n)}{64} < r(Q, n).$$

Hence for every b, $(b \geqq m_n)$, the point P_b belongs to $C(Q, n)$, which in turn is a subset of a domain of the set G_n. Let H_n be the set of all points P_b with $b \geqq m_n$, and let R_n be a domain of the set G_n containing H_n. By (c) of axiom 1′ there exists a point P common to all the sets $\overline{R}_1, \overline{R}_2, \overline{R}_3, \cdots$. It is easy to show that P is a sequential limit point of the sequence $P_1, P_2, P_3, \cdots$. Thus the space S is complete with respect to the distance function $\rho(x, y)$.

DUKE UNIVERSITY

EXAMPLES OF HARMONIC FUNCTIONS*

BY A. J. MARIA

1. *Introduction.* It is known that a function $u(r, \theta)$, harmonic in the unit circle and having the property that the absolute continuity of $\int_{\theta_1}^{\theta_2} |u(r, \theta)|\, d\theta$ is uniform with respect to $r<1$, is uniquely determined by the boundary values which it takes on nearly everywhere, for approach merely along radii. If, however, the less restrictive condition,

$$\int_0^{2\pi} |u(r, \theta)|\, d\theta < M$$

for $r<1$, is satisfied, then $u(r, \theta)$ is not necessarily uniquely determined by the boundary values which it takes on even in the wide sense nearly everywhere.† In fact, if infinite boundary values are allowed, two harmonic functions, $u_1(r, \theta)$ and $u_2(r, \theta)$, each having the property that the integral from 0 to 2π with respect to θ of its absolute value is bounded for $r<1$, can take on the same boundary values *everywhere* in the wide sense and yet fail to be identical. Such are, for instance, the two functions

$$\frac{a(1-r^2)}{1-2r\cos\theta+r^2}, \quad \frac{b(1-r^2)}{1-2r\cos\theta+r^2}, \quad a \neq b.$$

The harmonic function

$$\frac{\partial}{\partial\theta}\,\frac{1-r^2}{1-2r\cos\theta+r^2}$$

takes on the value zero along all radii of the unit circle.‡ On the other hand, a harmonic function which takes on continuously at each point of a circle a finite boundary value is uniquely determined by those boundary values. The facts are the same for three or more dimensions.

These results lead one to inquire if a harmonic function is unique when it takes on in the wide sense at each point of a

* Presented to the Society, September 10, 1931.

† G. C. Evans, *The Logarithmic Potential*, 1927, pp. 46–54.

‡ G. C. Evans, this Bulletin, vol. 37 (1931), p. 143.

circle or sphere, as the case may be, a given finite boundary value. It will be shown, by means of examples, that this is not necessarily true.

Another query, related to the above, concerns the Cauchy integral representation of a function, $f(z)$, analytic in the unit circle. The function $f(z)$ will be said to have a *normal* Cauchy integral representation if

$$f(z) = \frac{1}{2\pi i}\int_c \frac{u(t) + iv(t)}{t - z}\,dt, \tag{1}$$

where $u(t)$ and $v(t)$ are real, summable functions on the unit circle and $f(z)$ takes on the value $u(t)+iv(t)$ in the wide sense at nearly every point of the unit circle.* The question is the following: If $f(z)$ takes on, in the wide sense, summable boundary values $u(t)+iv(t)$ nearly everywhere, is it given by (1)? It will be shown that even if $f(z)$ takes on *continuous* boundary values in the wide sense *everywhere*, it is not necessarily given by (1).†

2. *Approach in the Wide Sense.* A variable point M, interior to a circle or sphere, is said to approach a fixed point P of a circle or sphere in the wide sense if there exists a positive constant $\alpha<\pi/2$ such that the angle θ, $-\pi/2<\theta<\pi/2$, between the ray PM and the interior normal at P satisfies the inequality $|\theta|<-\alpha+\pi/2$ as M approaches P.

A function $f(M)$ takes on a boundary value $h(P)$ at P in the wide sense if $\lim f(M)=h(P)$ when M, interior to the circle or sphere, approaches P in the wide sense. The function takes on the boundary value continuously if $\lim f(M)=h(P)$ when M, interior to the circle or sphere, approaches P in an arbitrary manner.

3. *A Function Harmonic in the Unit Circle which Approaches Zero at Every Point of the Circle in the Wide Sense and Is Not Identically Zero.* The simplest case is that where approach is merely in the wide sense at just one point and continuous elsewhere.

If there is a function, $u(x, y)$, harmonic in the unit circle,

* G. C. Evans, *The Logarithmic Potential*, 1927, p. 66.

† These problems were suggested to me by Professor G. C. Evans.

which approaches zero continuously at all points of the circle except at P and approaches zero in the wide sense at P, then the analytic function which has this harmonic function as its real part becomes, by means of a conformal transformation which carries the interior of the unit circle into the upper half plane and the point P into the infinite point, an entire function which has the property that it is pure imaginary along the real axis and that its real part approaches zero uniformly as the variable point approaches the infinite point in any sector, with vertex at the origin, whose sides are contained in the upper half plane. Conversely, the real part of an entire function having this property becomes, by means of a conformal transformation which carries the upper half plane into the interior of the unit circle, a harmonic function satisfying the given conditions.

It turns out that the entire function*

$$f(z) = u + iv = i\sum_{2}^{\infty} \frac{z^{n-2}}{\Gamma\left(1+\frac{1}{(\log n)^{a}}\right)}, \qquad (0 < a < 1),$$

has this property. In fact, Malmquist obtains the inequality

$$|f(z)| < r^{-\epsilon}M(\phi), \qquad \phi = \arg z, \qquad r = \operatorname{mod} z,$$

where $0<\epsilon<1$ and $M(\phi)$ may be taken as constant for all ϕ such that $0<\alpha<\phi<2\pi-\alpha$. In this inequality, ϵ is independent of a.

4. *The Three-Dimensional Case.* Let us define $u_1(x, y)$ and $v_1(x, y)$ as follows: $u(x, y-1)=u_1(x, y)$ and $v(x, y-1)=v_1(x, y)$. Obviously, $u_1(x, y)\cdot v_1(x, y)$ is harmonic, since it is, except for a constant factor, the imaginary part of $[u_1(x, y)+iv_1(x, y)]^2$. The function $U(x, y, z)=u_1(x, y)\cdot v_1(x, y)$ is a harmonic function in three dimensions.

It now follows from these definitions and the preceding paragraph that $U(x, y, z)$ vanishes at every finite point of the plane $y=1$ and satisfies the inequality

$$(1') \qquad |U(x, y, z)| < \rho^{-2\epsilon}[M(\phi)]^2,$$

where, in particular, ϵ may be chosen so that $1/2<\epsilon<1$, ρ being the distance from (0, 1, 0) to the projection of (x, y, z) on the

* J. Malmquist, Acta Mathematica, vol. 29 (1905), pp. 203–215.

plane $z=0$, and ϕ the angle which ρ makes with the positive axis of x. We may take $M(\phi)$ as a constant for all ϕ such that $0<\alpha<|\phi|<\pi-\alpha$.

If an inversion is made in the unit sphere, the space $y>1$ will go into the interior Σ of the sphere S of radius $1/2$ and center at $(0, 1/2, 0)$. The function

$$V(X, Y, Z) = \frac{1}{R} U\left(\frac{X}{R^2}, \frac{Y}{R^2}, \frac{Z}{R^2}\right),$$

where $R=(X^2+Y^2+Z^2)^{1/2}$, and where (X, Y, Z) is the point into which (x, y, z) is transformed by the inversion, is harmonic in Σ. It takes on the boundary value zero continuously at each point of S except $(0, 0, 0)$.

Consider now a point $M(X, Y, Z)$ in Σ which approaches $(0, 0, 0)$ in the wide sense. Since approach is in the wide sense and the inversion is conformal, the point (x, y, z) which corresponds to $M(X, Y, Z)$ will eventually lie within a right circular cone having as its axis the axis of y and vertical half-angle ϕ which satisfies the inequality $|\phi|<\pi/2-\alpha$, α being a positive number. The magnitude of the projection on the axis of y of the line segment having as its end points $(0, 1, 0)$ and (x, y, z) is not greater than ρ; the number ρ has been defined following formula (1′). It then follows that

$$r' \sin \alpha \leqq \rho,$$

where $r'=(x^2+(y-1)^2+z^2)^{1/2}$. Moreover, it is evident that we have $\lim r'/r=1$ as r becomes infinite, where $r=(x^2+y^2+z^2)^{1/2}$. From these facts, one finds that

$$|V(X, Y, Z)| = \frac{1}{R}|U(x, y, z)| < \frac{[M(\phi)]^2 r}{[\sin \alpha]^{2\epsilon} r'^{\epsilon}}$$

$$< \frac{2[M(\phi)]^2}{[\sin \alpha]^{2\epsilon}} [R^2]^{\epsilon-1/2}$$

for all points $M(X, Y, Z)$ which are sufficiently near $(0, 0, 0)$. This proves that $V(X, Y, Z)$ approaches zero in the wide sense at $(0, 0, 0)$.

5. *Normal Cauchy Integral Representation.* Consider the function

$$g(z) = \frac{f(z)}{ie^{f(z)/i}},$$

where $f(z)$ has been defined in §3. If the upper half plane is transformed conformally into the interior of the unit circle, $g(z)$ will be transformed into a function $l(z) = u(z) + iv(z)$, analytic in the interior of this circle. The function $l(z)$ takes on real values continuously on the circle except at one point where it takes on in the wide sense the value zero. In fact $f(z)/i$ is real along the real axis; it approaches zero uniformly as z approaches the infinite point in such a way that ϕ, the argument of z, satisfies the relation $0 < \alpha < \phi < 2\pi - \alpha$ (see paragraph 3). Hence $g(z)$ approaches zero uniformly as z approaches the infinite point subject to the above condition. It is further evident, since we know that $\lim f(z)/i = +\infty$ as z becomes infinite along the positive real axis, that these boundary values are continuous, and therefore, a fortiori, summable.

If $l(z)$ has a normal Cauchy integral representation, then the absolute continuity of $\int_{\theta_1}^{\theta_2} |l(z)|\,d\theta$ is uniform for $r < 1$.* Since $|v(r, \theta)| \leqq |l(r, \theta)|$, it follows that the absolute continuity of $\int_{\theta_1}^{\theta_2} |v(r, \theta)|\,d\theta$ is uniform for $r < 1$. This is a contradiction since $v(r, \theta)$ (see Introduction) would vanish identically and this would imply that $l(z)$ is a constant, which it is not.

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NOTE ON RELATIONS BETWEEN ELLIPTIC INTEGRALS AND SCHLÖMILCH SERIES

BY W. R. BENNETT

In a recent study of the problem† of calculating modulation products in the output of a rectifier when two sinusoidal waves of different frequencies are applied, a number of interesting and apparently novel mathematical relations were deduced as a by-product of the analysis. Unless otherwise noted, $0 \leqq b \leqq a$, $a > 0$, $a + b \leqq \pi$, and m and n may be any positive integers or zero. The following theorems are typical of the results obtained.

* Fichtenholz, Fundamenta Mathematicae, vol. 13 (1929), pp. 1–33.

† A paper discussing this problem and its solution has been prepared for publication elsewhere.

I. *If $m+n$ is odd, and $a+b<\pi$,*

$$\sum_{r=1}^{\infty} \frac{J_m[(2r-1)a]J_n[(2r-1)b]}{2r-1} = \frac{\Gamma\left(\frac{m+n}{2}\right)b^n}{4\Gamma(n+1)\Gamma\left(\frac{m-n+2}{2}\right)a^n} \cdot F\left(\frac{m+n}{2}, \frac{n-m}{2}; n+1; \frac{b^2}{a^2}\right).$$

II. *If $m+n$ is even, adding $\pi^2/16$ to the right-hand side for $m=n=0$, we have*

$$\sum_{r=1}^{\infty} \frac{J_m[(2r-1)a]J_n[(2r-1)b]}{(2r-1)^2} = \frac{\Gamma\left(\frac{m+n-1}{2}\right)b^n}{8\Gamma(n+1)\Gamma\left(\frac{m-n+3}{2}\right)a^{n-1}} \cdot F\left(\frac{m+n-1}{2}, \frac{n-m-1}{2}; n+1; \frac{b^2}{a^2}\right).$$

III. *If $m+n$ is odd, adding $\pi^2 a/16$ to the right-hand side for $m=1$, $n=0$, and $\pi^2 b/16$ for $m=0$, $n=1$, we have*

$$\sum_{r=1}^{\infty} \frac{J_m[(2r-1)a]J_n[(2r-1)b]}{(2r-1)^3} = \frac{\Gamma\left(\frac{m+n-2}{2}\right)b^n}{16\Gamma(n+1)\Gamma\left(\frac{m-n+4}{2}\right)a^{n-2}} \cdot F\left(\frac{m+n-2}{2}, \frac{n-m-2}{2}; n+1; \frac{b^2}{a^2}\right).$$

It would be possible to generalize the above results to make the exponent in the denominator on the left-hand side any positive integer, but the tabulation of the exceptional cases, which increase in number as this exponent increases, would prove rather cumbersome.

In the series appearing above, the arguments of the Bessel functions are proportional to the rank of the term, and the series could, therefore, be technically regarded as Schlömilch series with one of the two Bessel functions in each term to be regarded as a part of the coefficient. Since the two Bessel functions play similar parts in the series, however, it seems preferable to regard these series as a more general type than the ordinary Schlömilch form which has only one Bessel function in each term; in fact the above series could be regarded as a degenerate case of a double Schlömilch series bearing somewhat the same relation to the ordinary Schlömilch series as the double Fourier series does to the single Fourier series. Apparently, series of the above nature have not as yet been studied to any extent in mathematical literature.

It will be noted that the first two terms in the argument of the hypergeometric functions are odd multiples of one-half, and that the third term is an integer. It follows that by the use of the difference equations for the hypergeometric function, and the relations

$$K = \frac{\pi}{2} F(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2), \qquad E = \frac{\pi}{2} F(-\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2),$$

the hypergeometric functions may be reduced to finite combinations of complete elliptic integrals with modulus $k = b/a$. A few typical results are the following:

$$(1) \quad \sum_{r=1}^{\infty} \frac{J_1[(2r-1)a]J_0[(2r-1)b]}{2r-1} = \frac{E}{\pi};$$

$$(2) \quad \sum_{r=1}^{\infty} \frac{J_0[(2r-1)a]J_1[(2r-1)b]}{2r-1} = \frac{E-(1-k^2)K}{k\pi};$$

$$(3) \quad \sum_{r=1}^{\infty} \frac{J_0[(2r-1)a]J_0[(2r-1)b]}{(2r-1)^2} = \frac{\pi^2}{8} - \frac{a}{\pi}[2E-(1-k^2)K];$$

$$(4)\quad \sum_{r=1}^{\infty} \frac{J_1[(2r-1)a]J_1[(2r-1)b]}{(2r-1)^2} = \frac{a}{3\pi k}[(1+k^2)E - (1-k^2)K];$$

$$(5)\quad \sum_{r=1}^{\infty} \frac{J_1[(2r-1)a]J_0[(2r-1)b]}{(2r-1)^2} = \frac{\pi^2 a}{8} - \frac{a^2}{9\pi}[(7+k^2)E - 4(1-k^2)K];$$

$$(6)\quad \sum_{r=1}^{\infty} \frac{J_0[(2r-1)a]J_1[(2r-1)b]}{(2r-1)^2} = \frac{\pi^2 b}{8} - \frac{a^2}{9\pi k}[(1+7k^2)E - (1+3k^2)(1-k^2)K].$$

It may be remarked that when $b=0$, (3) reduces to a Schlömilch series summation which was set as a problem* in the Mathematical Tripos of 1898.

The method by which the above results were derived consists of a comparison of various forms of the general coefficient in the double Fourier series expansion of the periodic function of two variables,

$$(7)\quad f_r(x, y) = \begin{cases} -(P\cos x + Q\cos y)^r, & P\cos x + Q\cos y < 0 \\ 0, & P\cos x + Q\cos y = 0 \\ (P\cos x + Q\cos y)^r, & P\cos x + Q\cos y > 0 \end{cases},$$

where r is zero or a positive integer. For convenience we take P and Q as positive, and Q not greater than P. The function satisfies any one of several sets of sufficient conditions† for the expansion, which may be written:

$$(8)\quad f_r(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} C_{mn} e^{i(mx+ny)},$$

$$(9)\quad C_{mn} = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f_r(x, y) e^{-i(mx+ny)} dx dy.$$

Although the application is of no importance in the mathematical argument, it may be of some interest to point out that when two sinusoidal waves of different frequencies are applied to a rectifier, the output may be expressed either as ($P\cos x$

* Watson, *Theory of Bessel Functions*, p. 634.

† Hobson, *Theory of Functions of a Real Variable*, vol. 2, Chap. VIII.

$+Q \cos y)$ $f_0(x, y)$ or $f_1(x, y)$ for a full wave linear rectifier, as $(1/2)[f_1(x, y)+P \cos x+Q \cos y]$ for a half wave linear rectifier, and as $(1/2)[f_2(x, y)+(P \cos x+Q \cos y)^2]$ for a half-wave square law rectifier, provided that we assign the special values to x and y, $x=pt+\theta_p$, $y=qt+\theta_q$, t being the time, p, q the angular velocities, and θ_p, θ_q the phase angles.

If we evaluate the integral (9) directly by determining the boundaries of the region where $P \cos x+Q \cos y$ is positive and integrating accordingly, we obtain expressions involving complete elliptic integrals. Instead of proceeding in this way, we may make use of the following Fourier series expansions,

$$(10)\quad \frac{4}{\pi} \sum_{r=1}^{\infty} \frac{1}{2r-1} \sin \frac{(2r-1)\pi u}{c} = \left\{ \begin{array}{rl} -1, & -c < u < 0 \\ 0, & u = 0 \\ 1, & 0 < u < c \end{array} \right\};$$

$$(11)\quad -\frac{4c}{\pi^2} \sum_{r=1}^{\infty} \frac{1}{(2r-1)^2} \cos \frac{(2r-1)\pi u}{c} = \left\{ \begin{array}{rl} -u, & -c \leqq u \leqq 0 \\ u, & 0 \leqq u \leqq c \end{array} \right\};$$

$$(12)\quad cu - \frac{8c}{\pi^3} \sum_{r=1}^{\infty} \frac{1}{(2r-1)^3} \sin \frac{(2r-1)\pi u}{c} = \left\{ \begin{array}{rl} -u^2, & -c \leqq u \leqq 0 \\ u^2, & 0 \leqq u \leqq c \end{array} \right\}.$$

The left-hand members of (10)–(12) represent $f_0(x, y)$, $f_1(x, y)$, $f_2(x, y)$, respectively, if $u=P \cos x+Q \cos y$ and $c>P+Q$.

Substituting the resulting expressions in the integrand of (9), and interchanging the order of summation and integration (which may easily be justified), we obtain generalized Schlömilch series forms for C_{mn}. Formulas (1)–(6) may now be established by direct comparison with the elliptic integral expressions obtained for C_{mn} by the first method.

To prove the more general formulas I–III, consider the trigonometric integrals,

$$\frac{2}{\pi}\int_0^{\infty} \frac{\sin u\lambda}{\lambda} d\lambda, \quad \frac{2}{\pi}\int_0^{\infty} \frac{d}{d\lambda}(\cos u\lambda)\frac{d\lambda}{\lambda}, \quad \frac{2}{\pi}\int_0^{\infty} \frac{d^2}{d\lambda^2}(\sin u\lambda)\frac{d\lambda}{\lambda}.$$

It is clear that if we set $u=P\cos x+Q\cos y$, these integrals represent, respectively, $f_0(x, y)$, $f_1(x, y)$, $f_2(x, y)$. Inserting the resulting expressions in the integrand of (9), interchanging the order of the integrations so that the infinite integration is performed last,* and finally interchanging the order of differentiation and finite double integration, we get expressions for C_{mn} in terms of the infinite integral of Weber and Schafheitlin which is in turn reducible to a hypergeometric function.† Formulas I-III are then obtained by direct comparison with the results obtained by the first method.

It appears that by substituting different forms for the function $f_r(x, y)$ a wide variety of additional results may be obtained. For example, if we replace $f_r(x, y)$ by the simple function $P\cos x+Q\cos y$, we have $C_{10}=C_{-1,0}=P/2$, $C_{01}=C_{0,-1}=Q/2$, and $C_{mn}=0$ for all other values of m and n. But the Fourier series

$$\frac{2c}{\pi}\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r}\sin\frac{r\pi u}{c}=u, \qquad (-c<u<c), \tag{13}$$

represents $P\cos x+Q\cos y$ if $u=P\cos x+Q\cos y$ and $c>P+Q$. Calculating C_{mn} from the series and comparing, we find

$$\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r}J_m(ra)J_n(rb)=0, \tag{14}$$

provided $m+n$ is odd and greater than one, and a and b satisfy the same restrictions as specified for formulas I–III. For $m=1$, $n=0$, and $m=0$, $n=1$, the right-hand member becomes $a/4$ and $b/4$ respectively. Known forms of Schlömilch series which represent null functions may be deduced‡ by setting $b=0$.

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* Justification of the interchange is a bit delicate, but ample theorems are available. See, for example, Hobson, *Theory of Functions of a Real Variable*, vol. 2, p. 348.

† Watson, *Theory of Bessel Functions*, p. 401.

‡ See Watson, *Theory of Bessel Functions*, pp. 634–636.

ON CERTAIN DYNAMICAL SYSTEMS WITH POINTS OF PEANO

BY MARIE CHARPENTIER*

1. *Introduction.* Recently I have studied differential systems of the form

$$\frac{dx_i}{dt} = X_i(x_j), \qquad (i, j = 1, \cdots, n),$$

in which the point (x_i) is restricted to lie in a rectangular domain of x_i space while the X_i are merely taken to be continuous.† As is well known, there may exist in this case more than one integral curve through certain "points of Peano."

Such differential systems have an obvious physical interest, for they represent the simplest possible extension of the ordinary dynamical type, in which a certain indeterminacy of motion may exist; in the dynamical case the manifold M is closed instead of rectangular. A first fundamental question in this direction is as to whether there exists an analog of the central motions as defined by Birkhoff.‡

In the present paper, I have commenced with the case of most interest, in which there is uniqueness for decreasing t.§ An analogous set of central motions is defined in this case and these are determined for the sphere $(n=2)$.

2. *Central Motions with a Condition of Uniqueness for Decreasing t.* (a). Consider a point P at an instant t_0 on the manifold M. Let us define for any $\epsilon>0$, an open molecule σ of diam-

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† Bulletin des Sciences Mathématiques, vol. 54 (1930), p. 203. Comptes Rendus, vol. 191 (1930), p. 912; and vol. 192 (1931), p. 913. Bulletin International de l'Académie Polonaise, 1931, p. 191, and Mathematica (Cluj), vol. 5, pp. 65–99.

‡ See, in particular, *Dynamical Systems*, Colloquium Publications of this Society, vol. 9, p. 189; and G. D. Birkhoff and P. A. Smith, Journal de Mathématiques, 1928, p. 345.

§ That is, if we consider the curves of motion passing through P for $t=t_0$, they coincide for $t<t_0$. For the study of this case in the plane see p. 87 of my Mathematica paper.

eter ϵ, containing P. Each point of the molecule is supposed to move in all admissible directions as time increases, but, of course, these points will always fill a continuum.

In what follows we shall use the following fundamental property of the motions. Let $(P)_t$ be the set of points to any of which P may have moved after a time t, $(P)_t$ being by hypothesis a single point for $t<0$; for any point P, any $t_0>0$, and any $\epsilon>0$, there is a $\delta>0$ such that, if Q is within a distance δ of P, then any point of $(Q)_t$ is within ϵ of some point of $(P)_t$ for $|t| \leqq t_0$. It follows that any set $(P)_t$ is closed.

(b). Consider now the above defined molecule σ. If P is not an equilibrium point and if ϵ is small enough, σ moves outside its first position.

When it is possible to choose ϵ small enough so that σ never overlaps its first position again, we shall call P a *wandering point*. Let the set of all wandering points be called W. If σ overlaps itself when t has increased from t_0 to $t_0+\tau$, it has to do so when t has decreased from t_0 to $t_0-\tau$, and conversely.

Thus we arrive at the same set of points W if we let the time decrease instead of increase in the above definition. If P is a wandering point, so are all the points of σ, since σ is open; thus W is open, and the set $M_1=M-W$ is closed. For any point of M_1, any corresponding molecule σ will overlap itself after any period of time as t increases (and decreases), no matter how small the molecule may be.

If a given point P is in M_1, then so is all of the motion through P for decreasing t.

For consider any point $(P)_t$, $t<0$, and any molecule σ about $(P)_t$. Take a molecule σ' about P so small that $(\sigma')_t$ lies wholly in σ. Then σ', hence $(\sigma')_t$ and therefore σ, intersects itself after any period of time, and thus $(P)_t$ is in M_1. The same reasoning shows that any unique curve of motion lies wholly in W or in M_1.

Similarly we define M_2 as the set of non-wandering points with respect to M_1, M_3 as the set of non-wandering points with respect to M_2 and so forth. Thus we obtain an ordered set of closed sets M, M_1, M_2, $\cdots$, M_ω, $M_{\omega+1}$, $\cdots$, M_r. We shall call the limiting set, M_r, the set of *central motions* in analogy with the ordinary definition.

(c). By the customary argument it is possible to prove the following property.

When the time increases and decreases every wandering point approaches M_1 uniformly.

Furthermore if P is a point of M_1, it will either stay indefinitely in M_1, or leave M_1. In the latter case it becomes a wandering point and must nevertheless approach M_1 later. By the usual argument we obtain the following result. *The probability that an arc of a curve of motion lies inside of an arbitrary neighborhood of M_1 approaches unity uniformly as the interval of time for such an arc increases indefinitely.*

(d). Let us call, as usual, the limiting points of a curve of motion for increasing t, ω points, and its limiting points for decreasing t, α points.

The $\alpha(\omega)$ limit points A of a motion C are formed of complete motions. It is sufficient to show, for a fixed $t_0>0$, that if P is a point of A, then there is a point P' of $(P)_{t_0}$ in A; the property is evident for $t_0<0$. If Q is a point of C, then there are points $Q_{\tau_1}, Q_{\tau_2}, \cdots$ of C lying in $(Q)_{\tau_1}, (Q)_{\tau_2}, \cdots$ respectively and approaching P. The points $Q_{\tau_1+t_0}, Q_{\tau_2+t_0}, \cdots$ of C must then approach the point set $(P)_{t_0}$. They have a limit point P' in the set; thus P' is a point of A in $(P)_{t_0}$, as required.

The limit points of any set of complete motions consist of complete motions.

The proof is exactly as above except that the points $Q_{\tau_1}, Q_{\tau_2}, \cdots$ may lie on different motions. Instead of the sequence $M_1, M_2, M_3, \cdots$ above, used to define the central motions M_τ, we can use another set as follows. Let us call N_1 the set formed by the α limit points of all motions on M together with their geometrical limit points. It is easy to see that $M_1 \supset N_1$. Similarly we define N_2 on N_1, etc. The limiting set N_s consists of those motions which belong to their own α limit points, say E, and the geometrical limiting points of E.

(e). Evidently $N_s \subset M_\tau$; are they identical?* The set of central points contained in a small molecule σ must overlap itself in M_τ for $t_0<0$ however large t_0 and however small σ may be. Hence there exists in σ a pair of central points P_1 and Q_1, positions reached by the same point after a certain period of time. We shall assume that Q_1 precedes P_1.

Choose about P_1 a neighborhood σ_1 so small that both σ_1 and $(\sigma_1)_{t_1}$, the corresponding neighborhood of Q_1, together with

* See G. D. Birkhoff and P. A. Smith, loc. cit., p. 354.

their limit points shall be contained in σ. There exist in σ_1 two central points P_2 and Q_2 about which we can choose two molecules σ_2 and $(\sigma_2)_{t_2}$ completely interior to σ_1, Q_2 preceding P_2. Continue in this manner. We can choose the molecules σ_k such that

(a) the diameter of σ_k converges to zero as $k \to \infty$, and

(b) the periods of time t_k increase indefinitely.

Then the limiting point L of the molecules is interior to every molecule σ_k, and thus the curve of motion C through L for decreasing t has the following property (C belongs to M_r). For every k, $(L)_{t_k}$ has a point in σ_{k-1}; thus L is an α limit point of C. Moreover, as has been shown above, through L exists a complete curve of motion lying in the set of α limit points of C and hence lying in M_r and its own set of α limit points. Therefore we may make the following statement.

The complete motions lying in their own set of α limit points are everywhere dense on M_r. It readily follows that *through every central point passes a complete motion lying wholly in M_r.*

(f). It must be noticed that if a motion C has a common point with its α limit points and its ω limit points it does not necessarily belong to either one. However a motion lying in its set of α limit points and having a common point with its set of ω limit points is also part of the latter, and is *pseudo-recurrent* according to the usual definition.

The pseudo-recurrent motions and the recurrent motions, these being defined in the customary way, belong of course to M_r. Furthermore, in the present case, it is easy to see that M_1 and N_1 are not equivalent.

Take on the sphere a simple closed curve* C through an equilibrium point P, and let Q be a unique point of Peano on C. Let us assume that the motions through Q have as ω limit points a periodic motion C_1, for the motions lying on the right-hand side, P of course for C, and another periodic motion C_2 for the motions lying on the left-hand side. The point P is the only equilibrium point between C_1 and C_2. It is easy to see that PQ belongs to M_1, but not to N_1.

3. *Central Motions on the Sphere.* It is well known that if on

* This example was pointed out by H. Whitney; he also proved independently that M_r consists of complete motions.

the sphere a curve of motion C possesses the two following properties:

(1) C never crosses itself;

(2) on C there is a point P such that C for indefinitely increasing (or decreasing) t has always points arbitrarily near P; *then through P passes a periodic motion.*

It immediately follows from a theorem in §2 that such periodic points P are everywhere dense on the central motions, provided that we consider only the part of M_r which is not the set of equilibrium points. Thus we may make the following statement.

On a sphere M, the central motions M_r consist of the equilibrium points, the periodic motions, and their limit points.

These limit points form boundaries of regions; therefore the central motions which are not periodic terminate in equilibrium points or sets of equilibrium points and are, then, nowhere dense on M_r.

A periodic motion through a point P is the only central motion issuing from P since two periodic motions cannot intersect, and any other central motion, being terminated in equilibrium points for decreasing t, has no common points with a periodic motion. Moreover if P is a central, non-periodic point, there exist periodic points converging to P not lying on $H(P)$ (the set of curves of motion through P); thus there exist at most two different sequences of periodic motions converging to P, and thus to at most *two different central motions* through P. Evidently the recurrent and pseudo-recurrent motions are the equilibrium points and the periodic motions.

4. *Possibility of Choice.* Consider a point P and the set of the ω limit points of all the curves of motions through P, say $\Omega_\omega(P)$. If P_1 is a position reached by P at some later time, we have

$$\Omega_\omega(P) \supset \Omega_\omega(P_1).$$

The number of possible choices of P cannot increase; in particular, if P and P_1 are on the same periodic motion,

$$\Omega_\omega(P) = \Omega_\omega(P_1).$$

Consider now a point P on a central motion C on a sphere and suppose that, at P, the curve of motion is not unique. Let C_1 be

another curve of motion through P. Consider the arc P_1Q_1 of $(P)_\epsilon$, where P_1 lies on C and Q_1 on C_1. Since a motion passing through P_1Q_1 and not passing through P_1 or Q_1 cannot be a central motion, we define thus a small region PP_1Q_1 characteristic of the central motion C, or possibly of the two central motions C and C_1. Therefore we have the following result.

The central motions, except possibly the equilibrium points, from which may issue at least one other motion form at most a denumerable set.

If there exist two periodic motions C_1 and C_2 (or any two closed curves made up of motions of M) such that the region r bounded by C_1 and C_2 does not contain any equilibrium points in its interior, it is necessary, in order that some point P of r can choose to approach in the future either C_1 or C_2, that there be a unique periodic motion c in r.

If P is interior to r, c cannot coincide with C_1 or C_2. Moreover, P, as long as it stays on c, will be able to go to any point of r at almost any time, more precisely at a time T, $t-\tau \leqq T \leqq t+\tau$, t being arbitrarily chosen and τ the period on c. The above condition is also sufficient if c contains a point of Peano. To prove the first statement, take a point P_0 of r having two curves c_1 and c_2 asymptotic to C_1 and C_2 as t increases. Of course, the motion c through P_0 for decreasing t cannot be asymptotic to either C_1 or C_2. Let us draw two closed bands B_1 and B_2 of width ϵ including C_1 and C_2. Let P_1 and P_2 be the first points on c_1 and c_2 respectively in common with B_1 and B_2. By cutting the region R ($R=r-(B_1+B_2)+$boundary) by the arc $P_1P_0P_2$, we obtain a simply connected region which does not contain any equilibrium points; therefore c cannot be interior to this region. If we choose ϵ suitably, c has no common points with B_1 and B_2 and thus it has a common point Q with $P_1P_0P_2$. Hence following c further, we run into P_0 again, and obtain thereby a periodic motion. By the same argument, it is seen that through every point of r passes a motion issuing from P_0; thus c is the only periodic motion in r.

The motion c cuts r into two regions r_1 and r_2. Every point on c can pass to any point of r, while a point interior to, say, r_1, must be in B_1 after a certain interval of time, and can never reach a point of r_2. The sufficiency of the condition is not hard to prove.

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ON n-WEBS OF CURVES IN A PLANE

BY G. BOL

This note contains a proof of Theorem 4 of the list given by W. Blaschke* in a preceding paper.

If $t_i = const.$ represents n sheaves of curves in a plane, then the maximal number of linearly independent relations

$$\sum_i U_{ik}(t_i) = 0, \quad (k = 1, \cdots, m,\ i = 1, \cdots, n), \tag{1}$$

is

$$N = \tfrac{1}{2}(n-1)(n-2). \tag{2}$$

Let (1) be any set of such relations; then we consider $U_{ik}(t_i)$, $(k=1, \cdots, m)$, for a fixed i to be the m coordinates of a point describing a curve $p_i(t_i)$ in an affine m-space.

If we can prove that the curves $p_i(t_i)$ all lie in parallel linear subspaces of dimension N, our theorem is proved, for this means that between the coordinates of every p_i there exist linear relations with the same constant coefficients, which express $m-N$ of the coordinates in terms of the other N. And this means that of the m relations (1) there can be only N linearly independent.

If we assume our functions U_{ik} to be differentiable a suitable number of times, however, this last statement comes down to proving that among the vectors

$$\frac{d}{dt_i} p_i(t_i) = p_i'(t_i), \quad p_i''(t_i), \quad p_i'''(t_i), \cdots, \tag{3}$$

there cannot be more than N linearly independent ones.†

We will prove this for $n=5$, $N=6$; the proof can easily be extended to all values of n. To avoid the use of many indices, we will write (1) in the form

$$p_1(u) + p_2(v) + p_3(r) + p_4(s) + p_5(t) = 0. \tag{4}$$

* W. Blaschke, *Results and problems about n-webs of curves in a plane*, this Bulletin, vol. 38 (1932), p. 828.

† This does not really make it necessary to assume the functions (1) to be analytic; from a certain order m we can always replace (3) by an existence statement for solutions of a differential equation.

As our parameters t_i are given functions of the coordinates in the plane of our curves, and are all independent functions, we can express them as functions of u and v. We then differentiate the vector equation (3) with respect to u and v and find

$$
\begin{aligned}
(5)\quad & 0 = p_1' + p_3' r_u + p_4' s_u + p_5' t_u, \\
& 0 = p_2' + p_3' r_v + p_4' s_v + p_5' t_v; \\
(6)\quad & 0 = p_1'' + p_3'' r_u^2 + p_4'' s_u^2 + p_5'' t_u^2 + p_3' r_{uu} + p_4' s_{uu} \\
& \qquad + p_5' t_{uu}, \\
& 0 = p_2'' + p_3'' r_v^2 + p_4'' s_v^2 + p_5'' t_v^2 + p_3' r_{vv} + p_4' s_{vv} \\
& \qquad + p_5' t_{vv}; \\
(7)\quad & 0 = p_1''' + p_3''' r_u^3 + p_4''' s_u^3 + p_5''' t_u^3 + L(p_i'', p_i'), \\
& 0 = p_2''' + p_3''' r_v^3 + p_4''' s_v^3 + p_5''' t_v^3 + L(p_i'', p_i'); \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
(8)\quad & 0 = p_3'' r_u r_v + p_4'' s_u s_v + p_5'' t_u t_v + p_3' r_{uv} + p_4' s_{uv} + p_5' t_{uv}, \\
& 0 = p_3''' r_u^2 r_v + p_4''' s_u^2 s_v + p_5''' t_u^2 t_v + L(p_i'', p_i'); \\
(9)\quad & 0 = p_3''' r_u r_v^2 + p_4''' s_u s_v^2 + p_5''' t_u t_v^2 + L(p_i'', p_i'), \\
& 0 = p_3^{\mathrm{iv}} r_u^3 r_v + p_4^{\mathrm{iv}} s_u^3 s_v + p_5^{\mathrm{iv}} t_u^3 t_v + L(p_i''', p_i'', p_i'); \\
(10)\quad & 0 = p_3^{\mathrm{iv}} r_u^2 r_v^2 + p_4^{\mathrm{iv}} s_u^2 s_v^2 + p_5^{\mathrm{iv}} t_u^2 t_v^2 + L(p_i''', p_i'', p_i'), \\
& 0 = p_3^{\mathrm{iv}} r_u r_v^2 + p_4^{\mathrm{iv}} s_u s_v^3 + p_5^{\mathrm{iv}} t_u t_v^3 + L(p_i''', p_i'', p_i'); \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

Here L always means a linear combination of the vectors in brackets, and $i = 3, 4, 5$. In this way we get two groups of equations. The first expresses all the derivatives of p_1 and p_2 as combinations of those of p_3, p_4, and p_5. The latter can be used to prove that of these there can be no more than 6 linearly independent.

If we assume for a moment that the equations (8), (9), (10) are not in a disturbing way dependent, then the result is obvious. For then we can have at the utmost 3 independent vectors p_i', of the vectors p_i'' one can be expressed by means of the others and p_i' as a consequence of (8), so we get only two extra independent vectors, and (9) shows that vectors p_i''' can give only one extra dimension. The total number is exactly $3+2+1=6$.

So the only thing that remains to be proved is that relations (8), (9), (10) are really independent. Now in (8) the coefficients of p_i'' cannot vanish. For $r_u=0$ would mean that r was a function of u alone, and therefore that sheaves $r=$const. and $u=$const. would coincide. So (8) really gives us a relation between the p_i''. To show that (9) gives 2 relations we have to consider the matrix

$$\left\| \begin{matrix} r_u^2 r_v & s_u^2 s_v & t_u^2 t_v \\ r_u r_v^2 & s_u s_v^2 & t_u t_v^2 \end{matrix} \right\|$$

and show that it is of rank 2. But one of the determinants is

$$r_u r_v s_u s_v \cdot \begin{vmatrix} r_u & s_u \\ r_v & s_v \end{vmatrix}$$

and none of the factors can vanish, the last one since this would imply the dependence of the functions r and s, which is again impossible. Finally the essential determinant in (10) is equal to

$$r_u r_v s_u s_v t_u t_v \begin{vmatrix} r_u & s_u \\ r_v & s_v \end{vmatrix} \begin{vmatrix} s_u & t_u \\ s_v & t_v \end{vmatrix} \begin{vmatrix} r_u & t_u \\ r_v & t_v \end{vmatrix}$$

so that from (10) we can really compute p_i^{iv} as linear combinations of p_i''', p_i'', p_i'. We see that there is no danger for dependency of the equations, and our theorem is proved.

Of course if $n>5$, we have a similar proof, only the determinants we have to consider are of higher order. We find

$$N = n-2+n-3+n-4+\cdots+2+1 = \tfrac{1}{2}(n-1)(n-2).$$

As a corollary, for $n=4$, we have: *If a 2-dimensional surface in k-space can be generated in two different ways as a translation surface, it lies in a linear three-dimensional subspace.**

For the assumption leads to a vector equation (4) with $n=4$ and our formula gives $N=3$.

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* See S. Lie, Leipziger Berichte, 1897, p. 186.

ON THE DERIVATION OF NECESSARY CONDITIONS FOR THE PROBLEM OF BOLZA*

BY G. A. BLISS AND I. J. SCHOENBERG

1. *Introduction.* The problem of Bolza with variable end-points in the calculus of variations† is that of finding in a class of arcs

$$y_i = y_i(x), \quad (i = 1, 2, \cdots, n; x_1 \leqq x \leqq x_2), \tag{1}$$

satisfying the differential equations and end-conditions

$$\phi_\alpha(x, y, y') = 0, \qquad (\alpha = 1, 2, \cdots, m < n), \tag{2}$$

$$\psi_\mu[x_1, y(x_1), x_2, y(x_2)] = 0, \quad (\mu = 1, 2, \cdots, p \leqq 2n + 2), \tag{3}$$

one which minimizes an expression of the form

$$I = G[x_1, y(x_1), x_2, y(x_2)] + \int_{x_1}^{x_2} f(x, y, y')dx. \tag{4}$$

Morse and Myers‡ have recently studied this problem in a somewhat different form. They seek to find in the class of arcs (1) satisfying the differential equations (2) and the end-conditions

$$x_s = x_s(\alpha_1, \cdots, \alpha_r), \; y_{is} = y_{is}(\alpha_1, \cdots, \alpha_r), \qquad (s = 1, 2; i = 1, 2, \cdots, n), \tag{5}$$

one which minimizes a sum

$$I = \theta(\alpha_1, \cdots, \alpha_r) + \int_{x_1}^{x_2} f(x, y, y')dx. \tag{6}$$

It has been customary to assume that the matrix of derivatives of the functions ψ_μ in (3) with respect to the variables x_s,

* Presented to the Society, October 29, 1932.

† Bolza, *Über den "Anormalen Fall" beim Lagrangeschen und Mayerschen Problem mit gemischten Bedingungen und variablen Endpunkten*, Mathematische Annalen, vol. 74 (1913), pp. 430–446. Bliss, *The problem of Bolza in the calculus of variations*, Annals of Mathematics, (2), vol. 33 (1932), pp. 261–274. In the text here the latter paper will be referred to by the symbol B_2.

‡ *The problems of Lagrange and Mayer with variable end-points*, Proceedings of the American Academy of Arts and Sciences, vol. 66 (1931), pp. 235–253. In the text this paper will be designated by the symbol *MM*.

$y_{is}=y_i(x_s)$, $(s=1, 2; i=1, \cdots, n)$, is of rank p at the end-values of the particular minimizing arc E_{12} to be studied,* an assumption which insures the non-singularity of the end-values and the independence of the conditions (3). For the parametric form (5) Morse and Myers do not assume that the matrix of derivatives of the functions (5) is of rank r at the parameter values defining the ends of E_{12},† and hence the problem as formulated by them seems to have greater generality. In a footnote on page 244 of their paper, Morse and Myers call attention to this fact and make the statement that the necessary conditions for a minimum deduced by them from the first variation can not easily be derived from the results given by Bliss for the corresponding Lagrange problem.‡

In the following pages the authors wish to show how the methods used by Bliss for the Lagrange problem can be adapted with very slight modifications to deduce analogous theorems for the problem of Bolza. This will be done in §2 below. By the introduction of new variables, in §3, the parametric problem of Morse and Myers can be immediately reduced to one of the Bolza type. The necessary conditions usually derived from the first variation for the parametric form of the problem turn out to be simple corollaries of the corresponding theorems for the problem of Bolza.

The method used in deducing the theorem of §2 still applies when the function f in the expression (4) is identically zero. The problem is then that of Mayer with variable end-points as formulated by Bliss.§ The theorem deduced by Myers‖ for his modification of this problem is also an immediate consequence of the theorem of §2 below by the method of §3.

2. *The Problem of Bolza.* For a more precise formulation of the problem of Bolza see B_1, §§1 and 8, where the particular case

* Bolza, loc. cit., p. 433. Bliss, *The problem of Lagrange in the calculus of variations*, American Journal of Mathematics, vol. 52 (1930), condition (d) p. 690. We shall refer to the latter paper in the text by the symbol B_1.

† *MM*, pp. 236, footnote p. 244.

‡ B_1, §8, especially pp. 692, 693.

§ *The problem of Mayer with variable end-points*, Transactions of this Society, vol. 19 (1918), pp. 305–314.

‖ *Adjoint systems in the problem of Mayer under general end conditions*, this Bulletin, vol. 38 (1932), pp. 303–312, Theorem 4 of p. 311.

$G=0$ is investigated, and B_2, §§1, 2. In particular we make the four assumptions B_1, (a), (b), (c), and (d). Furthermore $G(x_1, y_{i1}, x_2, y_{i2})$ is supposed to be of class C' in a neighborhood of the end-points of E_{12}.

The additional term G in the expression I necessitates some slight changes in B_1, §8. Formula $B_1(45)$ becomes

$$\begin{aligned}\lambda_0 I_1(\xi, \eta) = &- \int_{x_1}^{x_2} \lambda_r \zeta_r dx - \lambda_0 f(x_1)\xi_1 - c_i\eta_i(x_1) + \lambda_0 f(x_2)\xi_2 \\ &+ \eta_i(x_2)F_{y_i'}(x_2) + \lambda_0 G_{x_1}\xi_1 + \lambda_0 G_{y_{i1}}[\eta_i(x_1) + y'_{i1}\xi_1] \\ &+ \lambda_0 G_{x_2}\xi_2 + \lambda_0 G_{y_{i2}}[\eta_i(x_2) + y'_{i2}\xi_2].\end{aligned}$$

The arguments following $B_1(45)$ hold without change and lead to the conclusion that there is a set of constants $\lambda_0, d_1, \cdots, d_p$, not all zero, such that the equation

$$\begin{aligned}0 = &- \int_{x_1}^{x_2} \lambda_r \zeta_r dx + [\lambda_0(-f(x_1) + G_{x_1} + G_{y_{i1}}y'_{i1}) \\ &+ d_\mu(\psi_{\mu x_1} + \psi_{\mu y_{i1}}y'_{i1})]\xi_1 \\ &+ [\lambda_0(f(x_2) + G_{x_2} + G_{y_{i2}}y'_{i2}) + d_\mu(\psi_{\mu x_2} + \psi_{\mu y_{i2}}y'_{i2})]\xi_2 \\ &+ [-c_i + \lambda_0 G_{y_{i1}} + d_\mu\psi_{\mu y_{i1}}]\eta_i(x_1) \\ &+ [F_{y_i'}(x_2) + \lambda_0 G_{y_{i2}} + d_\mu\psi_{\mu y_{i2}}]\eta_i(x_2)\end{aligned} \tag{7}$$

holds for every set of admissible variations ξ_1, ξ_2, $\eta_i(x)$ of the arc E_{12}. Choosing the constants c_i so as to make the coefficients of $\eta_i(x_1)$ disappear, we infer, from the fact that ζ_r, ξ_1, ξ_2, $\eta_i(x_2)$ can be chosen arbitrarily for admissible variations, that $\lambda_r \equiv 0$ and also that the matrix

$$\begin{Vmatrix} -F(x_1) + \lambda_0(G_{x_1} + G_{y_{i1}}y'_{i1}) & -F_{y_i'}(x_1) + \lambda_0 G_{y_{i1}} & & \\ \psi_{\mu x_1} + \psi_{\mu y_{i1}}y'_{i1} & \psi_{\mu y_{i1}} & & \\ & & F(x_2) + \lambda_0(G_{x_2} + G_{y_{i2}}y'_{i2}) & F_{y_i'}(x_2) + \lambda_0 G_{y_{i2}} \\ & & \psi_{\mu x_2} + \psi_{\mu y_{i2}}y'_{i2} & \psi_{\mu y_{i2}} \end{Vmatrix}$$

is of rank p. Hence also the matrix

$$\left\| \begin{matrix} -F(x_1) + F_{y_i'}(x_1)y_{i1}' + \lambda_0 G_{x_1} & -F_{y_i'}(x_1) + \lambda_0 G_{y_{i1}} \\ \psi_{\mu x_1} & \psi_{\mu y_{i1}} \\ F(x_2) - F_{y_i'}(x_2)y_{i2}' + \lambda_0 G_{x_2} & F_{y_i'}(x_2) + \lambda_0 G_{y_{i2}} \\ \psi_{\mu x_2} & \psi_{\mu y_{i2}} \end{matrix} \right\|$$

is of rank p, which is equivalent to the statement that the relation

$$[(F - F_{y_i'}y_i')dx + F_{y_i'}dy_i]_1^2 + \lambda_0 dG = 0 \tag{8}$$

is a consequence of the system of equations

$$d\psi_\mu \equiv \psi_{\mu x_1}dx_1 + \psi_{\mu y_{i1}}dy_{i1} + \psi_{\mu x_2}dx_2 + \psi_{\mu y_{i2}}dy_{i2} = 0, \quad (\mu = 1, 2, \cdots, p). \tag{9}$$

The multipliers $\lambda_0, \lambda_\alpha(x)$ can not all vanish simultaneously at a point $x=a$, since this would imply $B_i \equiv 0$ in equation (17) of paper B_1, and also $v_i(a)=0$, from B_1 (16). Hence B_1 (17) would imply $v_i \equiv 0$; in particular $v_i(x_1)=c_i=0$, and $v_i(x_2)=F_{y_i'}(x_2)=0$. From (7) we then find

$$d_\mu(\psi_{\mu x_1} + \psi_{\mu y_{i1}}y_{i1}') = 0, \qquad d_\mu\psi_{\mu y_{i1}} = 0,$$
$$d_\mu(\psi_{\mu x_2} + \psi_{\mu y_{i2}}y_{i2}') = 0, \qquad d_\mu\psi_{\mu y_{i2}} = 0,$$

from which we infer that $d_\mu = 0$, since the rank of the matrix of the coefficients is p. Hence $\lambda_0 = d_\mu = 0$, in contradiction to our previous result. We have thus proved the following theorem.

THEOREM 1. *For every minimizing arc E_{12} for the problem of Bolza there exist constants c_i and a function*

$$F = \lambda_0 f + \lambda_\alpha(x)\phi_\alpha \tag{10}$$

such that the equations

$$F_{y_i'} = \int_{x_1}^{x} F_{y_i}dx + c_i, \qquad \phi_\alpha = 0, \tag{11}$$

hold at every point of E_{12}, and furthermore such that the end-points of E_{12} satisfy, besides the equations (3), *the condition that* (8) *holds for every set of differentials $dx_1, dy_{i1}, dx_2, dy_{i2}$ which satisfy the equations* (9). *The first multiplier λ_0 is a constant. The multipliers $\lambda_\alpha(x)$ have definite limits $\lambda_\alpha(x-0), \lambda_\alpha(x+0)$ at every value of x*

on x_1x_2 and these limits are equal and the $\lambda_\alpha(x)$ are continuous except possibly at the values of x defining the corners of E_{12}. The elements of the set λ_0, $\lambda_\alpha(x)$ do not vanish simultaneously at any point of E_{12}.

The form of the expression I to be minimized has no bearing on questions regarding the normality of E_{12}; hence the results of B_1, §9 apply as well for the problem of Bolza.

3. *The Problem of Bolza in the Formulation of Morse and Myers.* In order to derive necessary conditions we suppose that a particular arc E_{12} of class D', whose end-points are given by (5) for $(\alpha)=(0)$, minimizes I in the class of all arcs (1) neighboring E_{12} and satisfying the system (2) and whose end-points are given by (5) for values (α) near the values (0).

The problem of Morse and Myers can be phrased as a problem of Bolza as follows. We adjoin to the set of functions $y_i(x)$ new functions $\alpha_h(x)$ satisfying the differential equations $\alpha_h'(x)=0$. The problem is then that of finding in the class of arcs

$$y_i = y_i(x), \qquad \alpha_h = \alpha_h(x), \tag{12}$$
$$(i = 1, \cdots, n; h = 1, \cdots, r; x_1 \leqq x \leqq x_2)$$

satisfying the differential equations and end-conditions

$$\phi_\beta(x, y, y') = 0, \qquad \alpha_h' = 0, \qquad (\beta = 1, 2, \cdots, m), \tag{13}$$
$$\psi_s = x_s - x_s(\alpha_1, \cdots, \alpha_r) = 0, \quad \psi_{is} = y_{is} - y_{is}(\alpha_1, \cdots, \alpha_r) = 0,$$

one which minimizes the expression I in (6). The particular minimizing arc E_{12}^* of the form (12), whose properties are to be studied, has $\alpha_h(x)\equiv 0$, $(h=1, \cdots, r)$. In the end-conditions (13) we think of the constants α_h as the initial values at $x=x_1$ of the functions $\alpha_h(x)$.

All of the assumptions of §2 are satisfied by this problem in the Bolza form. In particular the functional matrix of the end-conditions (13) has its determinant of derivatives with respect to the variables x_1, x_2, y_{i1}, y_{i2} equal to unity. Applying the result of §2 we know that there exist constants c_i, c_{n+h} and a function

$$F = \lambda_0 f + \lambda_\beta(x)\phi_\beta + \lambda_{m+h}(x)\alpha_h'$$

such that the equations

$$F_{y_i'} = \int_{x_1}^{x} F_{y_i} dx + c_i, \qquad F_{a_h'} = \lambda_{m+h} = c_{n+h},$$

hold along E_{12}^*, and furthermore such that at the ends of E_{12}^* the equation

$$(14) \quad [(F - y_i' F_{y_i'})dx + F_{y_i'} dy_i + \lambda_{m+h} d\alpha_h]_1^2 + \lambda_0 d\theta = 0$$

is a consequence of the equations

$$(15) \qquad dx_s - x_{sh} d\alpha_h = 0, \qquad dy_{is} - y_{ish} d\alpha_h = 0,$$

where x_{sh} and y_{ish} are notations for $\partial x_s/\partial \alpha_h$ and $\partial y_{is}/\partial \alpha_h$ at $(\alpha) = (0)$. The constants λ_{m+h} are all zero since in equation (14) they are coefficients of the differentials $d\alpha_{h2}$ on which no conditions are imposed by the equations (15) which contain only $d\alpha_{h1} = d\alpha_h$. We have therefore proved the following theorem.

THEOREM 2. *With the minimizing arc E_{12} there is associated a set of multipliers*

$$(16) \qquad \lambda_0, \lambda_\beta(x), \qquad (\beta = 1, 2, \cdots, m),$$

and a set of constants $c_1, \cdots, c_n$ such that the equations

$$F_{y_i'} = \int_{x_i}^{x} F_{y_i} dx + c_i$$

are satisfied at every point of E_{12}, where $F = \lambda_0 f + \lambda_\beta \phi_\beta$. Furthermore the equation

$$[(F - F_{y_i'} y_i')dx + F_{y_i'} dy_i]_1^2 + \lambda_0 d\theta = 0$$

is an identity in $d\alpha_h$ when dx_1, dx_2, dy_{i1}, dy_{i2} and $d\theta$ are evaluated for $(\alpha) = (0)$ and expressed in terms of the differentials $d\alpha_h$ by the equations (15). *The elements of* (16) *do not vanish simultaneously at any point of E_{12}.*

According to the definitions of Morse and Myers, and Bliss, the admissible arc E_{12} is normal (AB) (see *MM*, §5) if and only if the arc E_{12}^* is normal for the corresponding problem of Bolza (see B_1, §9). Criteria for normality are given in B_1, §9, and *MM*, §5. It can be shown now that Theorem 4 in *MM* follows immediately out of the condition of Bliss concerning the nonvanishing of the determinant $B_1(49)$. From (13), $B_1(44)$, and

the equations $\alpha_k' = 0$ we derive for a set of admissible variations along E_{12}*

$$\begin{aligned} \Psi_s(\xi, \eta, u) &= \xi_s - x_{sk}u_k, \\ \Psi_{is}(\xi, \eta, u) &= \eta_i(x_s) + y'_{is}\xi_s - y_{isk}u_k, \end{aligned} \tag{17}$$

where the constants u_k are the variations of the constant functions $\alpha_k(x)$. From these we find immediately

$$\Psi_{is} = \eta_i(x_s) - \bar{\eta}_{is} + y'_{is}\Psi_s \tag{18}$$

with the notation

$$\bar{\eta}_{is} = (y_{isk} - y'_{is}x_{sk})u_k$$

of Morse and Myers. A necessary and sufficient condition for E_{12}* to be normal, according to the definition of Bliss, is that there exist $2n+2$ sets of admissible variations $\xi_{s\mu}$, $\eta_{i\mu}(x)$, $u_{k\mu}$, $(\mu = 1, \cdots, 2n+2)$, such that the determinant

$$\begin{vmatrix} \Psi_s(\xi_\mu, \eta_\mu, u_\mu) \\ \Psi_{is}(\xi_\mu, \eta_\mu, u_\mu) \end{vmatrix}$$

is different from zero. The first equations (17), and (18), show that this determinant is identical with the determinant MM (5.1), with the notations ξ_s in place of γ_s. It follows also that Theorem 5 in MM is a corollary of the further results of Bliss in B_1, §9.

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PERIODIC SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

BY W. B. FITE

The present article contains a generalization and a correction of certain results given by the writer in an earlier article under the same title.* Consider the equation

$$(1) \qquad L(y) \equiv p_0 y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n y = 0,$$

in which the coefficients are one-valued and of period l, and $p_0 \neq 0$ for any real value of x. If $y_1, y_2, y_3, \cdots, y_n$ form a fundamental system of solutions, then

$$D \equiv \begin{vmatrix} y_1 & y_1' & \cdots & y_1^{(n-1)} \\ y_2 & y_2' & \cdots & y_2^{(n-1)} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ y_n & y_n' & \cdots & y_n^{(n-1)} \end{vmatrix} = D_0 e^{I(a,x)},$$

where

$$I(a, x) = -\int_a^x \frac{p_1}{p_0} dx,$$

and where $D_0 (\neq 0)$ is the value of D for $x = a$. Since the coefficients of (1) are of period l, $y_i(x+l)$, $(i = 1, 2, \cdots, n)$, form a fundamental system of solutions, and therefore

$$(2) \qquad y_i(x + l) = \sum_{j=1}^{n} a_{ij} y_j.$$

The characteristic equation of the substitution (2) is

$$(3) \qquad \begin{vmatrix} a_{11} - \omega & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \omega & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \omega \end{vmatrix} = 0.$$

* Fite, Annals of Mathematics, (2), vol. 28 (1926), pp. 59–64.

If

$$(\omega - \omega_1)^{\mu_1}, \quad (\omega - \omega_2)^{\mu_2}, \quad \cdots, \quad (\omega - \omega_k)^{\mu_k},$$

where $\mu_1+\mu_2+\cdots+\mu_k=n$, are the elementary divisors of the left member of (3), there is a canonical fundamental system of solutions $y_{i,j}$, $(i=1, 2, \cdots, k; j=1, 2, \cdots, \mu_i)$, such that*

$$\begin{aligned}
y_{i,1}(x+l) &= \omega_i y_{i,1}, \\
y_{i,2}(x+l) &= y_{i,1} + \omega_i y_{i,2}, \\
&\cdots\cdots\cdots \\
&\cdots\cdots\cdots \\
y_{i,\mu_i}(x+l) &= y_{i,\mu_i-1} + \omega_i y_{i,\mu_i}.
\end{aligned}$$

In terms of these solutions
$D(x+l)=$

$$\begin{vmatrix}
\omega_1 y_{11} & \omega_1 y_{11}' & \cdots & \omega_1 y_{11}^{(n-1)} \\
y_{11} + \omega_1 y_{12} & y_{11}' + \omega_1 y_{12}' & \cdots & y_{11}^{(n-1)} + \omega_1 y_{12}^{(n-1)} \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
y_{1,\mu_1-1} + \omega_1 y_{1,\mu_1} & y_{1,\mu_1-1}' + \omega_1 y_{1,\mu_1}' & \cdots & y_{1,\mu_1-1}^{(n-1)} + \omega_1 y_{1,\mu_1}^{(n-1)} \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots
\end{vmatrix}.$$

The first μ_1 rows are here indicated in detail. The remaining rows are formed similarly in sets of $\mu_2, \mu_3, \cdots, \mu_k$ each corresponding to the respective elementary divisors. It readily follows that

$$D(x+l) = \omega_1^{\mu_1}\omega_2^{\mu_2}\cdots\omega_k^{\mu_k} D(x).$$

But on the other hand

$$D(x+l) = D_0 e^{I(a,x+l)} = D_0 e^{I(a,x)+I(x,x+l)} = e^{I(x,x+l)},$$

where, as above,

$$I(u, v) = -\int_u^v \frac{p_1}{p_0} dx.$$

* See, for example, Horn, *Gewöhnliche Differentialgleichungen beliebiger Ordnung*, p. 85.

Hence

$$\omega_1^{\mu_1}\omega_2^{\mu_2}\cdots\omega_k^{\mu_k} = e^{I(x,x+l)}. \tag{4}$$

If there is a solution of (1) of period l, at least one of the ω's is 1; and conversely if at least one of the ω's is 1 there is a solution of (1) of period l. If all of the solutions of (1) are of period l, every $\omega=1$ and therefore

$$e^{I(x,x+l)} = 1, \quad \text{or} \quad \int_x^{x+l} \frac{p_1}{p_0}\,dx = 0.$$

But if every $\omega=1$ not every solution is necessarily of period l. For example, for certain values of a the Mathieu equation

$$y'' + (a - 2b\cos 2x)y = 0$$

has a solution of period π. We can take $l=\pi$. Moreover $p_0\equiv 1$ and $p_1\equiv 0$. Hence

$$\int_x^{x+l} \frac{p_1}{p_0}dx = 0.$$

That is, the product of the ω's is 1 and one of them is 1. Hence the other one is also 1. But not every solution is of period* π.

Consider now the adjoint of (1); namely

$$\bar{L}(z) = 0. \tag{5}$$

The functions

$$z_{ij} = \frac{1}{p_0}\frac{\partial \log D}{\partial y_{ij}^{(n-1)}} \tag{6}$$

form a fundamental set of solutions of† (5). It follows from relations (6) that when x is changed to $x+l$ the z's are transformed contravariantly to the y's. The determinant of this transformation is made up as follows.

Corresponding to the ith elementary divisor of (3) there is a minor of this determinant of the form

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* Ince, *Ordinary Differential Equations*, p. 177.

† Darboux, *Théorie des Surfaces*, vol. 2, p. 102.

$$\begin{vmatrix} \frac{1}{\omega_i} & -\frac{1}{\omega_i^2} & -\frac{1}{\omega_i^3} \cdots (-1)^{\mu_i-1}\frac{1}{\omega_i^{\mu_i}} \\ 0 & \frac{1}{\omega_i} & -\frac{1}{\omega_i^2} \cdots (-1)^{\mu_i-2}\frac{1}{\omega_i^{\mu_i-1}} \\ 0 & 0 & \frac{1}{\omega_i} \cdots (-1)^{\mu_i-3}\frac{1}{\omega_i^{\mu_i-2}} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ 0 & 0 & 0 \cdots \frac{1}{\omega_i} \end{vmatrix}.$$

The principal diagonal of this minor lies on the principal diagonal of the determinant, and every element of the determinant that does not belong to one of these minors is zero.

The roots of the characteristic equation of this transformation on the z's are the reciprocals of the roots of the corresponding equation for the y's. Moreover, the elementary divisors of the left member of the characteristic equation relating to the z's are the same as those for the left member of (3) with the exception that $\omega - 1/\omega_i$ takes the place of $\omega - \omega_i$. Hence (1) and its adjoint (5) have the same number of linearly independent solutions of period* l. It follows that if $L(y)$ is an exact derivative, the equation $L(y) = 0$ has a solution of period l, since $\overline{L}(z) = 0$ has a constant for a solution, and this is of period l.

If now (1) is self-adjoint the equation (3) is a reciprocal equation, and if n is odd there must be an odd number of roots that are their own reciprocals; that is, an odd number of roots must be ± 1. But if an odd number of these were equal to -1, the product of all the roots would be -1, since the other roots that are different from 1 occur in pairs of reciprocals. On the other hand, since (1) is by supposition self-adjoint, $p_1 = np_0'/2$, and

$$-\int_x^{x+l} \frac{p_1}{p_0}dx = -\frac{n}{2}\int_x^{x+l} \frac{p_0'}{p_0}dx = -\frac{n}{2}\log p_0\Big]_x^{x+l} = 0.$$

That is, by (4), the product of the roots of the characteristic equation (3) must be $+1$. Hence an odd number of these roots

* See Fite, loc. cit., p. 61.

are equal to 1 and equation (1) has at least one solution of period l.

A similar analysis applies if we pass to the complex plane and require merely that the coefficients be one-valued in the neighborhood of $z=z_0$ and that p_0 have a pole of order j at this point ($j \geqq 0$). If (1) is self-adjoint the product of the roots of the characteristic equation of the transformation on a fundamental system of solutions due to a circuit around $z=z_0$ is equal to

$$e^{I_C} = e^{nj\pi i} = \pm 1, \qquad I_C = -\frac{n}{2}\int_C \frac{p_0'}{p_0}dx,$$

according as nj is even or odd. If n is odd and j is even, nj is even and an odd number of the roots of the characteristic equation are equal to 1. Hence there is at least one solution that is one-valued in the neighborhood of $z=z_0$. But if n and j are both odd, an odd number of the roots of the characteristic equation are equal to -1, and there is at least one solution that is two-valued in the neighborhood of $z=z_0$.

If in (1) we again assume that the coefficients are real and of period l and put $y=e^{mx}t$, t will be a solution of the equation

$$(7)\quad p_0 t^{(n)} + (nmp_0 + p_1)t^{(n-1)} + \cdots + (m^n p_0 + m^{n-1}p_1 + \cdots + mp_{n-1} + p_n)t = 0.$$

If now t is a solution of (7) of period l, y is a solution of (1) such that

$$y(x+l) = e^{ml}y(x).$$

Hence (7) cannot have solutions of period l for more than r real values of m, where r is the number of distinct positive roots of (3). Nor can it have semi-periodic solutions with a given real multiplier σ for more than r real values of m, where r is the number of distinct roots of this characteristic equation that have the same sign as σ.

Suppose that

$$(8)\qquad L(y) = \phi(x),$$

where $\phi(x)$ is not identically zero and is of period l, has a solution y of period l. This assumption is certainly realized if (1) has no solution of period l other than* $y \equiv 0$. If (1) has r ($r>0$) linearly

* Fite, loc. cit., p. 60, Theorem I.

independent solutions of period l, equation (5) has the same number. We represent them by $z_1, z_2, \cdots, z_r$. If now in the Lagrange identity

$$\int [zL(y) - y\bar{L}(z)]dx = \psi(y, z)$$

we put for y a solution of (8) of period l and z_i for z, we get

$$\int z_i\phi(x)dx = \psi(y, z_i).$$

Now $\psi(y, z_i)$ is a polynomial in y and z_i and their derivatives together with the coefficients of $L(y)$ and their derivatives. It is therefore of period l and $z_i\phi(x)$ is the derivative of a function of this period. This means that in order that (8) have a solution of period l it is necessary that $z\phi(x)$ be the derivative of a function of this period in case z is any solution of (5) of this period.

We consider now whether this condition is sufficient. If t is a solution of the adjoint of the equation $L'(y)=0$, $tL'(y)$ is an exact derivative, as is also $tL'(y)+t'L(y)$. Hence $t'L(y)$ is an exact derivative and t' is a solution of $\bar{L}(y)=0$. Moreover, if t is a function such that $t'L(y)$ is an exact derivative, then $tL'(y)$ is an exact derivative and t is a solution of the adjoint of $L'(y)=0$. Hence this adjoint is $\bar{L}(t')=0$.

The solutions of $L(y)=0$ and $L(y)=c$, where c is any constant $\neq 0$, are solutions of $L'(y)=0$ and every solution of this last equation is a solution of one or the other of the two preceding ones. If, then, $L(y)=0$ has $r(r>0)$ linearly independent solutions of period l, and no more, and $L(y)=c$ has no such solution, $L'(y)=0$ has only r linearly independent solutions of this kind, and its adjoint has the same number. If now every one of the r linearly independent solutions $z_1, z_2, \cdots, z_r$ of $\bar{L}(y)=0$ of period l were the derivative of a function of this period, that is, if $t_i=\int z_i dx$, $(i=1, 2, \cdots, r)$, were of period l, then the t_i's together with $t_{n+1}\equiv 1$ would form a system of $r+1$ linearly independent solutions of the adjoint of $L'(y)=0$ of period l. But we have just seen that this adjoint has only r linearly independent solutions of this period in case $L(y)=0$ has only r linearly independent solutions of this kind and $L(y)=c$ has none. If $L(y)=0$ has no solution of period l other than $y\equiv 0$, the condition in

question is obviously satisfied and, as we have seen, $L(y)=c$ has a solution of this period. The condition is therefore sufficient in case $\phi(x)$ is a constant.

Suppose now that $\phi(x)$ is a non-constant function of period l. The solutions of (8) are also solutions of the equation

$$\frac{L(y)}{\phi(x)} = 1, \tag{9}$$

for all values of x for which $\phi(x)\neq 0$, and the coefficients of (9) are of period l. Moreover, $z_i\phi(x)$, $(i=1, 2, \cdots, n)$, form a linearly independent set of solutions of the adjoint of the equation $L(y)/\phi(x)=0$, and $z_1\phi(x)$, $z_2\phi(x)$, $\cdots$, $z_r\phi(x)$ are all of these that are of period l. Therefore a sufficient condition that (8) have a solution of period l is that $z_i\phi(x)$, $(i=1, 2, \cdots, r)$, be the derivatives of functions of period l, and this condition is necessary. This is equivalent to the condition $\int_x^{x+l} z_i\phi(x)dx=0$, $(i=1, 2, \cdots, r)$. We have therefore the following theorem.

THEOREM. *If $p_0(x)\neq 0$ and $\phi(x)\not\equiv 0$, while $\phi(x)$ and the coefficients of $L(y)$ are of period l, a necessary and sufficient condition that $L(y)=\phi(x)$ have a solution of period l is that $\int_x^{x+l} z\phi(x)dx=0$ when z is any solution of $\overline{L}(y)=0$ of period l.*

The sufficient condition given in Theorem 2 of the article referred to in the first paragraph is obviously incorrect, as may be seen by considering the equation

$$y^{iv} + y'' = 1.$$

The right member and the coefficients in the left member are of period 2π. Moreover $y=\sin x$ is a solution of period 2π of the corresponding homogeneous equation, which is self-adjoint, and $\int_x^{x+2\pi} \sin x\,dx=0$, while not every solution of the homogeneous equation is of period 2π. And yet the original equation has no periodic solution, since if y were such a solution we should have $y'''+y'=x+c$. But this is impossible since the left member would be periodic while the right member is not. This is in conformity with the conditions of the present theorem since $z=1$ is a solution of period 2π of the adjoint of the homogeneous equation and $\int_x^{x+2\pi} dx\neq 0$.

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TRANSFORMATIONS ASSOCIATED WITH THE LINES OF A CUBIC, QUADRATIC, OR LINEAR COMPLEX*

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1. *Introduction.* In this paper it is shown that two equations bilinear in p_{ik} and x_i define an extensive type of cubic complex and also map the complex on the space (x) so that each line is mapped by a point on itself. The cubic complex of lines joining corresponding points of the general cubic involutorial transformation is included as a special case. The method is also applied to two known cases of the quadratic complex and the linear non-special and special complex.

2. *The Cubic Complex.* Let

$$\sum x_i f_i(p) = 0, \tag{1}$$

and

$$\sum x_i F_i(p) = 0, \qquad (i = 1, 2, 3, 4), \tag{2}$$

be two equations bilinear in x_i and the line coordinates

$$p_{ik} = x_i y_k - x_k y_i.$$

The x_i and p_{ik} satisfy four identities of the type

$$x_i p_{jk} + x_j p_{ki} + x_k p_{ij} = 0. \tag{3}$$

The equations (1) and (2) represent two quadrics through (y) which meet in a C_4 through (y). The lines of the cubic cone with vertex (y) through C_4 belong to a cubic complex. If we eliminate the x_i from (1) and (2) and any two of (3), we have the equation of the cubic complex each line (l) of which is mapped by a point (x) on (l).

From (1), (2), and (3) we see that the p_{ik} are quartic functions of x_i. Hence any linear complex meets the cubic complex in a cubic congruence which is mapped by a quartic surface $F_4(x)$. Two linear complexes meet the cubic complex in a cubic ruled

* Presented to the Society, October 29, 1932.

surface which is mapped by the intersection of the two corresponding F_4's. Let

$$x_1 = x_2 = 0, \quad x_3 = x_4 = 0$$

be the pair of polar lines common to the linear complexes. If the coordinates of a line through a point (x) meeting the polar lines are substituted in (1) and (2), we have two cubic surfaces through $x_1 = x_2 = 0$, $x_3 = x_4 = 0$ which meet in a residual C_7, the variable intersection of the two F_4's. The C_7 meets each polar line in 4 points and is of genus 4. The common curve of all the F_4's meets C_7 in 22 points and is of genus 8. Hence the cubic complex is mapped on S_3 by the linear system of quartic surfaces through a C_9, $p = 8$.

3. *The Cubic Complex of the General Cubic Involutorial Transformation.* If (y) and (z) are conjugate points in the involutorial transformation they are polar conjugates with respect to three quadrics which by a suitable choice of coordinates have the equations*

$$\begin{aligned} a_1x_1x_4 + a_2x_2x_4 + a_3x_3x_4 - x_2x_3 &= 0, \\ b_1x_1x_4 + b_2x_2x_4 + b_3x_3x_4 - x_3x_1 &= 0, \\ c_1x_1x_4 + c_2x_2x_4 + c_3x_3x_4 - x_1x_2 &= 0. \end{aligned} \tag{4}$$

Hence, if

$$g_{ij} = y_iz_j + y_jz_i, \qquad (i, j = 1, \cdots, 4),$$

the three bilinear equations which define the involution may be written in the form

$$\begin{gathered} g_{23} = a_1g_{41} + a_2g_{42} + a_3g_{43}, \quad g_{31} = b_1g_{41} + b_2g_{42} + b_3g_{43}, \\ g_{12} = c_1g_{41} + c_2g_{42} + c_3g_{43}. \end{gathered} \tag{5}$$

The g_{ij} and p_{ik} satisfy the identities which express the fact that $(g_{i1}, g_{i2}, g_{i3}, g_{i4})$, $(i = 1, \cdots, 4)$, are on the line p_{ik}. Hence

$$g_{21}p_{34} + g_{23}p_{41} + g_{24}p_{13} = 0, \quad g_{12}p_{43} + g_{14}p_{32} + g_{13}p_{24} = 0. \tag{6}$$

If we substitute in (6) for g_{23}, g_{31}, g_{12} from (5) and use x_i for g_{4i}, $(i = 1, 2, 3, 4)$, then (6) is the form of (1) and (2) for this case. These two equations with any two of (3) define the cubic com-

* F. R. Sharpe and V. Snyder, *The* (1, 2) *correspondence associated with the cubic space involution of order two*, Transactions of this Society, vol. 25 (1923), pp. 1–12.

plex and map a line (l) of the complex on a point (x) of (l).* The cubic inversion† is the special case when

$$a_1 = b_2 = c_3 = 1,$$

and the other coefficients in (4) are zero.

4. *The First Type of the Quadratic Complex.* If the cubic complex of §2 reduces to a quadratic complex, then C_4 must break up into a cubic curve through (y) and either a line joining (y) to a fixed point O, or a fixed line (l).

In the first case the quadratic complex contains the bundle of lines through O, and the Kummer surface is a Steiner surface having three double lines through the triple point O. If the planes through the lines are

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0,$$

the quadratic complex has an equation of the form

$$ap_{23}p_{14} + bp_{31}p_{24} + cp_{12}p_{34} + F_2 = 0, \tag{7}$$

where F_2 is quadratic in p_{23}, p_{31}, p_{12}.

Consider the bilinear equation

$$p_{23}(Ax) + p_{31}(Bx) + p_{12}(Cx) = 0. \tag{8}$$

From equations (3), (7), and (8) we can derive the equations

$$\frac{p_{23}}{x_2(Cx) - x_3(Bx)} = \frac{p_{31}}{x_3(Ax) - x_1(Cx)} = \frac{p_{12}}{x_1(Bx) - x_2(Ax)},$$

which may be written

$$\frac{p_{23}}{f_1} = \frac{p_{31}}{f_2} = \frac{p_{12}}{f_3}, \tag{9}$$

where $f_i = 0$ is a quadric through a cubic C_3 which passes through O. Using (9) we find

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* For another method of mapping the cubic complex see D. Montesano, *Su di un complesso di rette del terzo grado*, Bologna Memoria, 1893, pp. 549–577. See p. 565.

† L. Godeaux, *Recherches sur les surfaces algébriques de genres zéro et de bigenre un*, Académie Royale de Belgique, Classe des Sciences, Bulletin, (5), vol. 12 (1926), pp. 892–904. See pp. 896–897.

$$
\begin{aligned}
p_{23} &= f_1(ax_1f_1 + bx_2f_2 + cx_3f_3),\\
p_{31} &= f_2(ax_1f_1 + bx_2f_2 + cx_3f_3),\\
p_{12} &= f_3(ax_1f_1 + bx_2f_2 + cx_3f_3),\\
p_{14} &= x_1F_2(f) - (b-c)f_2f_3x_4,\\
p_{24} &= x_2F_2(f) - (c-a)f_3f_1x_4,\\
p_{34} &= x_3F_2(f) - (a-b)f_1f_2x_4.
\end{aligned} \tag{10}
$$

Hence if (x) is given, the p_{ik} are quintic functions of the x_i. Conversely, if the p_{ik} are given satisfying (7), we can write (8) in the form

$$d_1x_1 + d_2x_2 + d_3x_3 + d_4x_4 = 0,$$

and, using (3), we can find for x_i the expressions

$$x_i = d_1p_{i1} + d_2p_{i2} + d_3p_{i3} + d_4p_{i4}, \qquad (i = 1, 2, 3, 4), \tag{11}$$

which are quadratic in the p_{ik}. The quintic surfaces (10), $p_{ik}=0$, have C_3 for double curve. From (10) we see that if p_{23}, p_{31}, p_{12} are fixed, then the lines of the complex lie in the fixed plane

$$x_1f_1 + x_2f_2 + x_3f_3 = 0, \tag{12}$$

and pass through the point (y) on the Steiner surface, where

$$
\begin{aligned}
y_1 &= (b-c)f_2f_3, \qquad y_2 = (c-a)f_3f_1,\\
y_3 &= (a-b)f_1f_2, \qquad y_4 = F_2(f).
\end{aligned} \tag{13}
$$

The equations (13) map the Steiner surface on a plane (f_1, f_2, f_3). If we substitute the values of (y) from (13) in (8) we have the condition that the point (y) lies on the line in which (8) meets (12). This is a cubic relation in (f) so that on the plane (f) we have a cubic curve of genus 1 to which corresponds a C_6, $p=1$, on the Steiner surface meeting C_3 in the 9 points apart from O in which C_3 meets the Steiner surface. This C_6 lies on all the quintic surfaces $p_{ik}=0$. The intersection of the quadratic complex with a linear complex is therefore mapped by a quintic surface $F_5 : C_3^2C_6$. Two quintic surfaces meet in a variable C_7, $p=1$, meeting C_3 in 11 points and C_6 in 9 points. Three quintic surfaces meet in 4 variable points.

5. *The Cremona Transformation Associated with the Quadratic Complex.* Consider a second bilinear equation

$$(14)\quad p_{23}(A'x) + p_{31}(B'x) + p_{12}(C'x) = d_1' x_1 + d_2' x_2 + d_3' x_3 + d_4' x_4 = 0.$$

Solving as in (11) for x_i in terms of p_{ik} and substituting from (10), we have expressions which are linear in the $F_5 : C_3^2 C_6$ with coefficients linear in (f). Hence we have a Cremona transformation of order 7 of the form $F_7 : C_3^3 C_6$. A surface $F_7 : C_3^3$ can be mapped on a plane by $C_5 : 18A$, quintics through 18 fixed points. The curve C_3 is mapped by $C_9 : 18A^2$ and the intersection of a variable $F_7 : C_3^3$ by $C_8 : 18A$. For the system $F_7 : C_3^3 C_6$, the $C_8 : 18A$ consist of $C_2 : 3A$, image of the variable curve C_7, $p = 0$, a fixed $C_3 : 9A$, image of C_6, $p = 1$, and a fixed curve C_9, $p = 1$. The triple curve C_3 meets C_6 and C_9 in 9 and 15 points, respectively, and C_7 in 12 points, C_6 and C_9 meet in 9 points and meet C_7 in 6 points.*

6. *The Involutorial Transformation belonging to the Quadratic Complex.* If we replace (14) by

$$(15)\quad p_{23}H_1 + p_{31}H_2 + p_{12}H_3 = 0,$$

where $H_i = 0$ is a quadric, the surfaces $f_i = 0$ are cubics through a C_7, $p = 5$, which passes through O, and the surfaces $p_{ik} = 0$ are of the form $F_7 : C_7^2$. The line in which the plane (8) met the plane (12) is replaced by a conic. If the point (y) is on the conic, we have a relation of the fifth order in f_1, f_2, f_3 so that the surfaces F_7 are of the form $F_7 : C_7^2 C_{10}$, $p = 6$. The curve C_7 meets the Steiner surface in 25 points apart from O. Hence C_7 meets C_{10} in 25 points.† To a line of the complex correspond two points (x) on the line. Two surfaces $F_7 : C_7^2 C_{10}$ meet in a variable C_{11}, $p = 8$, meeting C_7 and C_{10} in 27 and 15 points, respectively. Three of the surfaces meet in 8 variable points. Given a plane $(kx) = 0$, we can find the x_i in terms of the k_i and p_{ik} as in (11). Substituting these values for the (x_i) in (15) and for the p_{ik} from (10), we have a relation which is quadratic in the k_i of which (kx) is a factor. The other factor is the image of $(kx) = 0$ in the involutorial transformation which interchanges the two points (x)

* Compare D. Montesano, *Su le trasformazioni univoche dello spazio che determinano complessi quadratice di rette*, Reale Istituto Lombardo Rendiconti, (2), vol. 25 (1892). See p. 803.

† Compare D. Montesano, Reale Istituto Lombardo Rendiconti, (2), vol. 25 (1892), p. 802.

on a line p_{ik} of the complex. This involutorial transformation is therefore of the form $F_{16}:C_7^5C_{10}^2$. There are 25 trisecants of C_7 which meet C_{10} so that the surfaces all contain these 25 parasitic lines.

7. *The Second Type of the Quadratic Complex.* If the quadrics (7) and (8) are replaced by

$$ax_1 = bx_2,\ cx_1 = dx_2, \tag{16}$$

where a, b, c, and d are linear in the p_{ik}, the lines belong to the quadratic complex $ad-bc=0$. The intersection of this complex with a linear complex is mapped by a quartic surface $F_4:l^2$, where $l\equiv x_1=x_2=0$. The intersection with a linear congruence is mapped by the intersection of two cubic surfaces through l and the directrices of the congruence and is therefore a variable C_6, $p=1$, meeting l in four points. Two of the surfaces $F_4:l^2$ meet therefore in a fixed C_6, $p=1$, meeting l in four points. Consider a second pair of equations

$$ax_3 = bx_4, \qquad cx_3 = dx_4, \tag{17}$$

which give a second mapping of $ad-bc=0$. The two mappings determine a (5, 5) Cremona transformation $F_5:l^2C_6$. There is therefore an additional simple basis curve C_5, $p=0$, meeting l and C_6 in four and eight points, respectively.*

8. *The Second Type of Involutorial Transformation.* If we replace (16) by

$$aH_1 = bH_2, \qquad cH_1 = dH_2, \tag{18}$$

where $H_1=0$, $H_2=0$ are quadrics meeting in C_4, $p=1$, the intersection of the quadratic complex $ad-bc=0$ with a linear complex is mapped by a sextic surface $F_6:C_4^2$, and with a linear congruence by the intersection of two quartic surfaces through the directrices of the congruence and through C_4, that is, by a C_{10}, $p=7$, meeting C_4 in 16 points. Two surfaces $F_6:C_4^2$ meet therefore in a variable C_{10}, $p=7$, and a fixed C_{10}, $p=7$, meeting C_4 in 16 points. On each line of the complex are two associated points (x) and (x'). Given a plane $\Sigma a_ix_i'=0$, we find

* F. R. Sharpe, *Involutions of order n with an (n−2)-fold line*, Annals of Mathematics, (2), vol. 31 (1930), pp. 637–640.

$$x_1' = a_2p_{12} + a_3p_{13} + a_4p_{14}$$

and three similar equations. From (18) we have

$$H_1'H_2 - H_2'H_1 = 0.$$

Substituting for the x_i' and for the p_{ik} in terms of the x_i we have a relation which is quadratic in a_i and has Σa_ix_i for a factor. The other factor is the image of $\Sigma a_ix_i' = 0$ by the involutorial transformation defined by (18). Hence the transformation is of the form $F_{13}: C_4^5C_{10}^2$. There are 16 bisecants of C_4 which are bisecants of C_{10} and therefore lie on the F_{13}.*

9. *The Linear Complex.* Consider the linear complex $p_{12} = p_{34}$ and the bilinear equation

$$m_2p_{13} + m_3p_{14} + m_4p_{23} + m_5p_{24} + m_6p_{34} = 0,$$

where the m_i are linear in x_i. Proceeding as in §2 we may show that the linear complex is mapped on S_3 by the linear system of cubic surfaces through a C_5, $p = 1$.

The transformations belonging to a linear or special linear complex have been discussed synthetically by Montesano and Pieri. If we use one equation bilinear in x_i and p_{ik} and either $p_{12} = p_{34}$ or $p_{12} = 0$, we can readily obtain the results of Montesano† and Pieri.‡

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* D. Montesano, Reale Istituto Lombardo Rendiconti, (2), vol. 25 (1892), p. 803.

† Compare D. Montesano, Napoli Accademia delle Scienze Fisiche e Matematiche, Rendiconti, (2), vol. 2 (1888), pp. 181–188. D. Montesano, Rendiconti dei Lincei, vol. 4 (1st semester), 1888, pp. 207–215, 277–285.

‡ Pieri, Rendiconti di Palermo, vol. 6 (1892), pp. 234–244.

NOTE ON THE GRADIENT OF THE GREEN'S FUNCTION*

BY G. C. EVANS

1. *Integral of the Gradient of the Green's Function.* The author, in 1929,† mentioned without proof the fact that for a bounded open simply connected plane region T the integral

$$I(M) = \int_T |\nabla g(M, P)| \, d\sigma_P,$$

where $\nabla g(M, P)$ denotes the gradient of the Green's function with pole at M, is bounded, independently of the position of M in T. A corresponding result holds for convex regions in three dimensions. It is the purpose of this note to bring out some of the differences between the two- and three-dimensional situations by a comparison of the analyses for the general plane simply connected region and the star-shaped three-dimensional region.

THEOREM 1. *Let T be a bounded open simply-connected region in the plane, $g(M, P)$ its Green's function with pole at M, and $I(M) = \int_T |\nabla g(M, P)| d\sigma_P$. Then*

$$I(M) < (2\pi Tg)^{1/2}(1 - e^{-g})^{-1}, \tag{1}$$

where g is an arbitrary positive number and T stands for meas. T; *in particular*

$$I(M) < 2.216(\pi T)^{1/2}. \tag{1'}$$

For a given M the level curves of $g(M, P)$ are simple closed analytic curves, and neither $g(M, P)$ nor its gradient vanishes at a point of T. Denote by $h(M, P)$ the function conjugate to the Green's function, by ds the element of arc of the level curve s of $g(M, P)$, and by dn the element of arc of the normal trajectory n, taken as positive in the direction towards M. The integral $I(M)$ is evidently convergent.

The Jacobian $J = d(x, y)/d(g, h)$ has the value

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* Presented to the Society, August 31, 1932.

† Evans, *Discontinuous boundary value problems of the first kind for Poisson's equation*, American Journal of Mathematics, vol. 51 (1929), pp. 1–18.

$$(2)\qquad \frac{d(x,\,y)}{d(g,\,h)} = \left(\frac{dn}{dg}\right)^2 = \left(\frac{ds}{dh}\right)^2 = |\,\nabla g(M,\,P)\,|^{-2},$$

and if $u+vi=w(z)=e^{-(g+hi)}$ transforms the region T into the interior of the unit circle in the w-plane, the value of the Jacobian $J=d(x,\,y)/d(u,\,v)$ is

$$(3)\qquad \frac{d(x,\,y)}{d(u,\,v)} = \left|\frac{dz}{dw}\right|^2 = \left(\frac{dn}{dg}\right)^2 |\,w\,|^{-2}$$

for $dw/dz=-wd(g+hi)/dz=-w(\partial g/\partial x-i\partial g/\partial y)$, and $dg/dn=|dw/dz\,|/|w\,|$.

Hence if γ is any positive number, $\rho_1=e^{-\gamma}$, and ρ,θ are polar coordinates in the w-plane,

$$(4)\qquad I = \int_{(\rho<1)} \frac{1}{|\,w\,|}\left|\frac{dz}{dw}\right| du\,dw = \int_0^1 d\rho \int_0^{2\pi} \left|\frac{dz}{dw}\right| d\theta$$
$$= I_1 + I_2,$$

where

$$(4')\qquad I_1 = \int_{(\rho<\gamma)} \left|\frac{dg}{dn}\right| d\sigma = \int_{\rho_1}^1 f(\rho)\,d\rho,$$
$$I_2 = \int_{(\rho<\gamma)} \left|\frac{dg}{dn}\right| d\sigma = \int_0^{\rho_1} f(\rho)\,d\rho,$$

with

$$(4'')\qquad f(\rho) = \int_0^{2\pi} \left|\frac{dz}{dw}\right| d\theta.$$

Now by Schwarz's integral inequality, using (2), and writing T for meas. T, we have

$$I_1^2 \leqq \int_{(\rho<\gamma)} \left(\frac{dg}{dn}\right)^2 d\sigma \cdot \int_{(\rho<\gamma)} d\sigma < T \int_{(\rho<\gamma)} dg\,dh < 2\pi T\gamma,$$

and

$$(5)\qquad I_1 < (2\pi T\gamma)^{1/2}.$$

But by Hardy's well known theorem $f(\rho)$ is a non-decreasing function of ρ, for dz/dw is holomorphic in $|w\,|<1$. Hence

$$I_1 \geqq f(\rho_1)(1-\rho_1),\ I_2 \leqq f(\rho_1)\rho_1.$$

Consequently

$$f(\rho_1) \leqq \frac{I_1}{1-\rho_1} < \frac{(2\pi T\gamma)^{1/2}}{1-\rho_1},$$

and

$$(5') \qquad I_2 < (2\pi T\gamma)^{1/2}\frac{\rho_1}{1-\rho_1} = (2\pi T\gamma)^{1/2}\frac{e^{-\gamma}}{1-e^{-\gamma}}.$$

From (5) and (5′) we have (1), denoting γ by g.

In particular, we may choose g so as to make the right-hand member of (1) a minimum, for this expression is positive and becomes infinite as g tends to 0 or ∞. In fact, the derivative of $g^{1/2}(1-e^{-g})^{-1}$ vanishes at one and only one positive value of g, corresponding to the positive solution of the equation $e^g = 1+2g$. We calculate

$$g = 1.256,\ \rho_1 = 3.512^{-1} = .285,$$

whence we have (1′). This completes the proof of the theorem.

In particular, if T is a circle and M is at the center, we have $I = 2(\pi T)^{1/2}$, by direct calculation, a result which is near enough to (1′) to suggest an interesting problem in the calculus of variations. For the circle of unit radius, M being distant a, $a<1$, from the center, we have $I = \pi[(1-a^2)/a]\ \log\ [(1+a)/(1-a)]$, and in fact I is maximum when $a=0$.

2. *Corollaries.* The following remarks follow as corollaries. The integral $I(M)$, from (2), takes the forms

$$(6) \qquad I(M) = \int_0^\infty s_g dg = \int_0^{2\pi} n_h dh,$$

so that again, *each of these quantities is bounded, independently of the position of* M, in accordance with the theorem.

Moreover s_g is an increasing function of $\rho = e^{-g}$.* In fact, $s_g = \int |dz/dw|\, d(\rho\theta) = \rho f(\rho)$, which is strictly increasing, since $f(\rho)$ is non-decreasing. From (5) therefore, $gs_g < I_1 < (2\pi Tg)^{1/2}$ and

* W. Seidel, *Über die Ränderzuordnung bei konformen Abbildungen*, Mathematische Annalen, vol. 104 (1931), pp. 182–243.

$$s_\sigma < \left(\frac{2\pi T}{g}\right)^{1/2} = \left(\frac{2\pi T}{-\log \rho}\right)^{-1/2}. \tag{7}$$

Similarly, from the inequality on $f(\rho)$, we have the sharper inequality

$$s_\sigma < (2\pi T g)^{1/2}(e^\sigma - 1)^{-1}. \tag{7'}$$

Finally, $f(\rho) = 2\pi s_\sigma / c_\rho$, where c_ρ is the circumference of the circle of radius $\rho = e^{-\sigma}$, and therefore

$$I = 2\pi \int_0^1 \frac{s_\sigma}{c_\rho} d\rho, \tag{8}$$

where s_σ / c_ρ is a non-decreasing function of ρ.

3. *Three-Dimensional Case.* For a three-dimensional region T, which is homeomorphic with the interior of a sphere, the function $g(M, P)$ is defined by the methods of Harnack, but its gradient may vanish at some interior point even if the boundary of T is normal with respect to the Dirichlet problem.† The level surfaces would not then necessarily be homeomorphic with the surface of a sphere. For the theorem which follows, we restrict ourselves merely, however, to a region T, *bounded and homeomorphic with the interior of a sphere.*

THEOREM 2. *Let F be the set of points M of T from which T is star-shaped. Then F is null or closed with respect to T. Let $I(M) = \int_T |\nabla g(M, P)| d\tau_P$. Then for all points of F*

$$I \leqq 4.76(\pi^2 T)^{1/3} \leqq 4\pi g^{-1} + (4\pi g T)^{1/2}, \tag{9}$$

where g is an arbitrary positive number.

In particular, if T is convex, $F \equiv T$ and (9) holds for all points of T.

We prove that F is closed with respect to T by proving that the complementary set in T, namely $T - F$, is open. Let M be a point of $T - F$. Then, by definition, there is a point P of T such that the segment MP contains a point, say Q, not in T. Consider the lines through Q, and let τ_M, τ_P be spherical regions,

* J. J. Gergen, *Mapping of a three dimensional region on a sphere*, American Journal of Mathematics, vol. 52 (1930), pp. 197–224.

† J. J. Gergen, *Note on the Green function of a star-shaped three dimensional region*, American Journal of Mathematics, vol. 53 (1931) pp. 746–752.

with centers M and P and radii ρ_M and ρ_P, respectively, small enough to lie entirely in T. Given ρ_P we can choose ρ_M small enough so that every line through Q and a point of τ_M passes through a point of τ_P. This will be the case if $\rho_M < \rho_P \cdot (MQ/QP)$. Hence every point of $T-F$ is an interior point of $T-F$.

4. *Proof of Second Part of Theorem* 2. In order to prove the remainder of the theorem, let T' be a bounded simply-connected region whose boundary is normal with respect to the Dirichlet problem, and let M be a point of T' such that $\nabla g_{T'}(M, P)$ does not vanish for P in T'; this will be the case, in particular, if T' is star-shaped from M. Such a region may be mapped on the interior of a unit sphere in a one-one manner in such a way that the level surface $g(M, P) = g$ goes into the concentric sphere of radius $\rho = 1/(1+g)$, and the normal trajectories to the level surfaces go into the radii of the sphere; the mapping is not conformal, but on the other hand, again unlike the plane case, it reduces to the identical transformation as ρ tends to zero.*

Hence if we consider the cone of trajectories abutting on an element dS_g of the g-surface, which subtends a solid angle $d\omega$ at M and the corresponding element of surface dC_ρ on the sphere of radius $\rho = 1/(1+g)$, we have

$$\frac{dg}{dn}\,dS_g = d\omega = -\frac{dg}{d\rho}\,dC_\rho; \qquad \frac{dS_g}{dC_\rho} = -\frac{dn}{d\rho},$$

instead of $dS_g/dC_\rho = (dn/d\rho)^2$, which would be the case if the transformation were conformal. We have

$$(10)\qquad J = \frac{d(n, S_g)}{d(g, \omega)} = \left(\frac{dn}{dg}\right)^2,$$

$$J_1 = \left|\frac{d(n, S_g)}{d(\rho, \omega)}\right| = \frac{1}{\rho^2}\left(\frac{dn}{dg}\right)^2 = \frac{1}{\rho^2}\,\frac{dn}{dg}\,\frac{dS_g}{d\omega},$$

and

$$(11)\qquad I(M) = \int_{T'} \frac{dg}{dn}\,d\tau = \int_{(0<g<\infty)} \frac{dn}{dg}\,dg\,d\omega = 4\pi \int_0^1 \frac{S_g}{C_\rho}\,d\rho,$$

* J. J. Gergen, American Journal of Mathematics, vol. 52, loc. cit. The identity $(dg/dn)dS_g = d\omega$ is established by Gergen in the as yet unpublished part of his Thesis (Rice Institute, 1928). With the mapping already set up the demonstration is not difficult.

with similar forms for the two integrals $I_1=\int_{(g<\gamma)}(dg/dn)\,d\tau$ and $I_2=\int_{(g>\gamma)}(dg/dn)\,d\tau$, γ being a positive number.

Analogously to the plane case, we have

$$I_1^2 < T'\int_{(g<\gamma)}\left(\frac{dg}{dn}\right)^2 d\sigma = T'\int_{(g<\gamma)} dg\,d\omega = 4\pi\gamma T';$$

consequently

$$I_1 < (4\pi\gamma T)^{1/2}. \tag{12}$$

But for the treatment of I_2 we cannot follow the earlier method, since $f(\rho)=4\pi S_g/C_\rho$ is not now necessarily a non-decreasing function of ρ.

In fact, if T' is a sphere of radius a and $r=(x^2+y^2+z^2)^{1/2}$, M being at the origin, we have $g(M,P)=1/r-1/a$, and corresponding values of r and ρ are given by the relation $1/r-1/a=1/\rho-1$, so that

$$\frac{S_g}{C_\rho}=\frac{r^2}{\rho^2}=\left(1+\frac{1-a}{a}\rho\right)^{-2},$$

which is an increasing function of ρ if $a>1$, but a decreasing function if $a<1$. In the plane, the corresponding expression for two circles is independent of ρ.

Let us suppose, however, that T' is star-shaped from M. In this case the g-surfaces are homeomorphic with spherical surfaces, are analytic and enclose regions which are star-shaped from M; moreover*

$$|\nabla g(M,P)| > g/r, \qquad r=MP.$$

But $g(M,P)=1/r-v$, where v is a harmonic function positive in T', so that $1/r=v+g>g$; consequently

$$|\nabla g(M,P)| > g^2.$$

It follows, then, that

$$I_2=\int_{(g>\gamma)}\frac{dg}{dn}\,dT=\int_{(g>\gamma)}\left(\frac{dg}{dn}\right)^{-1}d\omega dg$$

$$<4\pi\int_\gamma^\infty g^{-2}dg=4\pi\gamma^{-1}.$$

* J. J. Gergen, American Journal of Mathematics, vol. 53, loc. cit.

This, with the inequality already obtained for I_1, gives

$$I < 4\pi\gamma^{-1} + (4\pi\gamma T')^{1/2}, \qquad \gamma > 0,$$

for all points M from which T' is star-shaped. The minimum value of this expression occurs for $\gamma = 2(2\pi/T)^{1/2}$, from which the value in the second member of (9) is obtained, namely, $I < 3(4\pi^2 T')^{1/3}$.

We note, incidentally, that from the values of dS_g and dn, already given, we obtain

$$I(M) = \int_0^\infty S_g dg = \int_{(4\pi)} N_\omega d\omega,$$

S_g being the total area of the g-surface and N_ω the total length of the trajectory originating at M in the angular element $d\omega$.

5. *Boundary of T not Normal.* We may pass now to the theorem for T, whose boundary may not be normal for the Dirichlet problem. It may be remarked, nevertheless, that the presence of an irregular boundary point very much restricts the nature of the set F.

Let M be a point of F, $r = MP$, Q the point where the ray r cuts the unit spherical surface with center M, P' the first point where it cuts the closed boundary set $\overline{T} - T$. Then (following Gergen) $r' = MP'$ is a lower semi-continuous function of Q on the unit sphere, $r' = f(Q)$. We have, then, $f(Q) = \lim_{n=\infty} f_n(Q)$, where the $f_n(Q)$ may be chosen as a strictly increasing sequence of continuous functions; in fact, the choice may be made so that we have $f_{n+1}(Q) - f_n(Q) > 1/n - 1/(n+1) = 1/(n^2+n)$. For n great enough the regions $T_n : r < f_n(Q)$ contain M and are (evidently) star-shaped from M. But each $f_n(Q)$ may be replaced by a function $\phi_n(Q)$, as regular as we please, such that uniformly $|f_n(Q) - \phi_n(Q)| \leqq 1/(2n^2+2n)$; in fact, such a function $\phi_n(Q)$ may be obtained by averaging $f_n(Q)$, more than once if necessary, over a circular neighborhood of Q on the unit sphere, the radius of the neighborhood being a constant δ chosen suitably small.*

Thus we obtain a strictly increasing sequence $\phi_n(Q)$ which defines a sequence of regions T_n', such that $\overline{T}_n'$ is contained in

* For the properties of approximations in average consult H. E. Bray, *Proof of a formula for an area*, this Bulletin, vol. 29 (1923), pp. 264–270.

T'_{n-1}. These regions are star-shaped from M, have boundaries which are regular with respect to the Dirichlet problem and, for n great enough, contain any given closed subset of T. In fact, $\lim_{n=\infty}\phi_n(Q)=f(Q)$. If $g_n(M, P)$ denotes the Green's function for T'_n, we have, uniformly in any closed subset K of $T-M$,

$$\lim_{n=\infty} g_n(M, P) = g(M, P),$$

$$\lim_{n=\infty} |\nabla g_n(M, P)| = \nabla g(M, P), \qquad P \text{ in } K.$$

But for n large enough so that K is contained in T'_n, we have

$$\int_K |\nabla g_n(M, P)|\, d\tau_P < 4\pi\gamma^{-1} + (4\pi\gamma T'_n)^{1/2} < 4\pi\gamma^{-1} + (4\pi\gamma T)^{1/2},$$

γ an arbitrary positive number. Hence

$$\int_K |\nabla g(M, P)|\, d\tau_P \leqq 4\pi\gamma^{-1} + (4\pi\gamma T)^{1/2},$$

and finally,

$$\int_T |\nabla g(M, P)|\, d\tau_P \leqq 4\pi\gamma^{-1} + (4\pi\gamma T)^{1/2}.$$

This completes the proof of Theorem 2.

For the sphere of radius R, with M at the center, we calculate

$$I = 4\pi R = 2(6\pi^2 T)^{1/3} = 3.63(\pi^2 T)^{1/3}.$$

THE RICE INSTITUTE

ON THE APPROXIMATE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS WITH BOUNDARY CONDITIONS*

BY W. H. Mc EWEN

In a recent paper† the writer has studied the convergence of trigonometric and polynomial approximating sums to the solution of the mth order linear differential system

$$L(y) \equiv \frac{d^m y}{dx^m} + Q_1(x)\frac{d^{m-1}y}{dx^{m-1}} + \cdots + Q_m(x)y = R(x),$$

$$U_i(y) \equiv \sum_{j=1}^{m}\left\{\alpha_i^{(j-1)}y^{(j-1)}(a) + \beta_i^{(j-1)}y^{(j-1)}(b)\right\} = h_i, \qquad (i = 1, 2, \cdots, m), \tag{1}$$

the approximating sums being defined by a least rth power criterion of best approximation. Thus in the polynomial case the approximating sum $P_n(x)$ was defined to be a polynomial of the nth degree which satisfies the boundary conditions $U_i(P_n) = h_i$ and at the same time minimizes the integral $\int_a^b |L(P_n) - R|^r dx$ in comparison with all other polynomials of that type, r being a preassigned constant >0.

The purpose of this paper is to discuss the convergence question when a different criterion is used to define $P_n(x)$, namely, $P_n(x)$ is *the approximating polynomial* of the nth degree for the solution of the system (1) if it minimizes the expression

$$\int_a^b |L(P_n) - R|^r dx + \sum_{i=1}^{m} C_i |U_i(P_n) - h_i|^{r_i} \tag{2}$$

in comparison with all other polynomials of like degree, the r's and C's being given constants >0. Kryloff‡ and Picone§ have

* Presented to the Society, August 31, 1932.

† W. H. McEwen, *Problems of closest approximation connected with the solution of linear differential equations*, Transactions of this Society, vol. 33 (1931), pp. 979–997.

‡ N. Kryloff, *Les Problèmes Fondamentaux de la Physique Mathématique et de la Science d'Ingénieur*, Monographie dans le Domaine de la Mathématique Appliquée, 1932, pp. 234–240.

§ M. Picone, *Sul metodo delle minime potenze ponderate e sul metodo di Ritz*, etc., Rendiconti di Palermo, vol. 52 (1928), pp. 237–244.

considered problems of minima similar to this but only for differential systems of the second order and with the additional restrictions, in the case of the former, that the r's be all alike and not less than 2, and in the case of the latter that they be all alike and not less than 1. The methods used here are essentially different from those of the authors cited.

The fact that for each value of n a polynomial $P_n(x)$ as defined above does exist, and when r and the r_i's are >1 is unique, can be proved without difficulty by means of an argument which has been used elsewhere* in establishing theorems on existence and uniqueness relating to other problems of minima. Without dwelling further upon this point we shall proceed at once to a discussion of the convergence question.

In regard to the differential system we shall assume that it has a unique solution $y(x)$, or in other words that the corresponding reduced system is incompatible. It is understood, as in the paper first cited, that $R(x)$ and $Q_1(x), \cdots, Q_m(x)$ are continuous. Then, by Theorem E of that paper, there exist polynomials $p_n(x)$, each of degree indicated by its subscript, satisfying the boundary conditions $U_i(p_n)=h_i$, and such that for $a \leqq x \leqq b$

$$(3) \qquad \left| y^{(k)}(x) - p_n^{(k)}(x) \right| \leqq \epsilon_n, \qquad (k = 0, 1, \cdots, m),$$

with $\lim_{n\to\infty} \epsilon_n = 0$. For the convergence proof when $r<1$, we shall assume that $y(x)$ has further properties of regularity, which will automatically be realized if the coefficients in the differential equation satisfy suitable conditions as to continuity and existence of derivatives, so that according to the remark following Theorem F in the earlier paper it is possible in (3) to make $\lim_{n\to\infty} n^{2/r}\, \epsilon_n^{\,s} = 0$, where s is a preassigned positive number. Under these hypotheses we shall prove that the m quantities $\left|y^{(k)}(x) - P_n^{(k)}(x)\right|$, $(k=0, 1, \cdots, (m-1))$, (and when $r<1$ the corresponding difference of the mth derivatives also) will converge uniformly to zero on the interval (a, b) as n becomes infinite.

Let $F(x)=y(x)-p_n(x)$, where $p_n(x)$ is the polynomial described in the paragraph above. Then $F(x)$ satisfies the m homogeneous boundary equations $U_i(F)=0$. Let $L(F)=Y(x)$, and let $\pi_n(x)$ be the approximating polynomial of the nth degree

* See, for example, W. H. McEwen, loc. cit., pp. 981–982.

defined with respect to the differential system $L(F)=Y$, $U_i(F)=0$, of which $F(x)$ is the unique solution. If γ denotes the expression which in this case corresponds to (2), then

$$\gamma = \int_a^b \left| L(\pi_n) - Y \right|^r dx + \sum_{i=1}^{m} C_i \left| U_i(\pi_n) \right|^{r_i} \tag{4}$$

is a minimum for polynomials of the nth degree. But 0 may be regarded as a particular polynomial of the nth degree and moreover $U_i(0)=0$. Hence it follows that

$$\gamma \leqq \int_a^b \left| Y \right|^r dx = \int_a^b \left| L(F) \right|^r dx \leqq K\epsilon_n^r, \tag{5}$$

where K is a constant expressible in terms of the upper bounds which can be placed on the coefficients $Q_i(x), \cdots, Q_m(x)$ occurring in the left-hand side of the differential equation.

Since each term of (4) is $\geqq 0$, it follows that each is $\leqq \gamma$. Hence

$$\int_a^b \left| L(\pi_n) - Y \right|^r dx = \int_a^b \left| L(\pi_n - F) \right|^r dx \leqq \gamma,$$

$$\left| U_i(\pi_n) \right| = \left(\frac{\gamma}{C_i}\right)^{1/r_i}, \qquad (i = 1, 2, \cdots, m).$$

By virtue of (5) these inequalities can be written

$$\begin{aligned} &\int_a^b \left| L(F - \pi_n) \right|^r dx \leqq K\epsilon_n^r, \\ &\left| U_i(\pi_n) \right| = \left(\frac{K}{C_i}\right)^{1/r_i} \epsilon_n^s, \quad (i = 1, 2, \cdots, m), \end{aligned} \tag{6}$$

where s stands for the least of the positive numbers $1, r/r_1, r/r_2, \cdots, r/r_m$, and ϵ_n is assumed to be <1.

From this point on we consider separately the two cases $r \geqq 1$ and $r<1$.

The case $r \geqq 1$. Let $u(x)=F(x)-\pi_n(x)$ and $Z(x)=L(u)$. Then $u(x)$ satisfies the m boundary equations $U_i(u)=-U_i(\pi_n)$, and hence is the unique solution of the differential system

$$L(u) = Z(x), \quad U_i(u) = -U_i(\pi_n), \qquad (i = 1, 2, \cdots, m).$$

Let $G(x, \xi)$ be the Green's function of the corresponding reduced system, and let $G_1(x), G_2(x), \cdots, G_m(x)$ be the respective solutions of the m systems

$$L(y) = 0, \quad U_i(y)\begin{cases} = 0, \text{ when } i \neq j, \\ = 1, \text{ when } i = j, \end{cases} \qquad (i, j = 1, 2, \cdots, m),$$

Then

$$u^{(k)}(x) = \int_a^b \frac{\partial^k}{\partial x^k} G(x, \xi) Z(\xi) d\xi - \sum_{i=1}^{m} G_i^{(k)}(x) U_i(\pi_n),$$
$$(k = 0, 1, \cdots, (m-1)).$$

Hence, by virtue of Hölder's inequality* and the fact that the functions $\partial^k G(x, \xi)/\partial x^k, G_i^{(k)}(x)$ are bounded on (a, b), we may write

$$| u^{(k)}(x) | \leqq \overline{G}\left\{ (b-a)^{(r-1)/r}\left[\int_a^b |Z(\xi)|^r d\xi \right]^{1/r} + \sum_{i=1}^{m} | U_i(\pi_n) | \right\},$$

where $\overline{G}$ is an upper bound of these functions.

Making use of (6), and the assumption made earlier that $\epsilon_n < 1$ so that ϵ_n^s is the greatest of the numbers $\epsilon_n, \epsilon_n^{r/r_i}$, we can write

$$| u^{(k)}(x) | \leqq K' \epsilon_n^s,$$

where

$$K' = \overline{G}\left\{ (b-a)^{(r-1)/r} K^{1/r} + \sum_{i=1}^{m} (K/C_i)^{1/r_i} \right\}.$$

But since $\pi_n(x)$ and $P_n(x)$ are the respective approximating polynomials for $F(x)$ and $y(x)$, two functions whose difference is the polynomial $p_n(x)$, it is clear that

$$u(x) = F(x) - \pi_n(x) = y(x) - P_n(x),$$

and consequently

$$| y^{(k)}(x) - P_n^{(k)}(x) | \leqq K' \epsilon_n^s,$$

* We use this result in the form which states that if $\phi(x) \geqq 0$ and r is any real number $\geqq 1$ then

$$\int_a^b \phi(x) dx \leqq (b-a)^{(r-1)/r} \left[\int_a^b \{\phi(x)\}^r dx \right]^{1/r}.$$

It is at this point that we use the hypothesis in the definition of $P_n(x)$ that $r \geqq 1$.

for $k=0, 1, \cdots, (m-1)$ and for all values of x on (a, b). Thus the convergence of the approximating process is assured, since $\lim_{n\to\infty} \epsilon_n = 0$.

*The case $r<1$.** For the treatment of this case we shall establish the following result.

AUXILIARY THEOREM. *If the polynomial $\pi_n(x)$ satisfies the m boundary equations*

$$U_i(\pi_n) = g_i, \qquad (i = 1, 2, \cdots, m)$$

and if $\delta = \max \left| L(\pi_n) \right|$ on $a \leqq x \leqq b$, then

$$\text{(a)} \quad \left| \pi_n^{(k)}(x) \right| \leqq C\left[\delta(b-a) + \sum_{i=1}^{m} \left| g_i \right|\right], (k = 0, 1, \cdots, m),$$

and if $Q_1(x), \cdots, Q_m(x)$ have bounded first derivatives,

$$\text{(b)} \quad \left| \frac{d}{dx} L(\pi_n) \right| \leqq Dn^2\left[\delta(b-a) + \sum_{i=1}^{m} \left| g_i \right|\right]$$

for all values of x on (a, b), C and D being constants independent of n and independent of the coefficients in $\pi_n(x)$.

To prove conclusion (a) we proceed as follows. Let $Z(x) = L(\pi_n)$. Then $\pi_n(x)$ is the unique solution of

$$L(y) = Z(x), \quad U_i(y) = g_i, \qquad (i = 1, 2, \cdots, m),$$

and hence

$$\pi_n^{(k)}(x) = \int_a^b \frac{\partial^k}{\partial x^k} G(x, \xi) Z(\xi) d\xi + \sum_{i=1}^{m} G_i^{(k)}(x) g_i, \qquad (k = 0, 1, \cdots, (m-1)),$$

where the G's are the same functions as defined in the first case. Hence

* The method of proof used in the first case is obviously not applicable when $r<1$ as it depended on a direct use of Hölder's inequality. The proof which we shall give here, however, can be applied equally when $r \geqq 1$, although the hypotheses which it imposes on the coefficients of the differential equation in order to insure a specified rate of convergence are somewhat more severe then in the first instance. As a point in its favor, however, it will be noted that the second method proves the convergence of m derivatives as compared with $(m-1)$ by the first method.

$$|\pi_n^{(k)}(x)| \leqq \bar{G}\left\{\int_a^b |Z(\xi)|\,d\xi + \sum_{i=1}^{m}|g_i|\right\}$$

$$\leqq \bar{G}\left\{\delta(b-a) + \sum_{i=1}^{m}|g_i|\right\}.$$

This holds for $k=0, 1, \cdots, (m-1)$, but since

$$\pi_n^{(m)}(x) = L(\pi_n) - Q_1\pi_n^{(m-1)} - \cdots - Q_m\pi_n,$$

a similar inequality holds for $k=m$. Hence there must exist a constant C as stated in the theorem such that

$$|\pi_n^{(k)}(x)| \leqq C\left[\delta(b-a) + \sum_{i=1}^{m}|g_i|\right]$$

for $k=0, 1, \cdots, m$ and for all values of x on (a, b).

To prove conclusion (b) we note that $\pi_n^{(m)}$ is a polynomial of the nth degree (in which the powers of x from $n-m+1$ on are absent), and therefore by Markoff's theorem on the derivative of a polynomial

$$|\pi_n^{(m+1)}(x)| \leqq n^2C\left[\delta(b-a) + \sum_{i=1}^{m}|g_i|\right].$$

Hence

$$\left|\frac{d}{dx}L(\pi_n)\right| = |\pi_n^{(m+1)} + Q_1\pi_n^{(m)} + \cdots + Q_m\pi_n'$$

$$+ Q_1'\pi_n^{(m-1)} + \cdots + Q_m'\pi_n|$$

$$\leqq Dn^2\left[\delta(b-a) + \sum_{i=1}^{m}|g_i|\right]$$

for values of x on (a, b), where D is a constant of the character specified in the theorem.

To proceed with the proof of the convergence theorem, let us suppose for the time being that

$$\epsilon_n^* < \delta/[4(1+m)M], \tag{7}$$

where M is an upper bound of the coefficients in $L(y)$. Let x_1 be a point of the interval (a, b) at which $L(\pi_n)$ attains its maximum value δ. Then, by virtue of the mean value theorem and conclusion (b) of the auxiliary theorem,

$$| L(\pi_n(x)) - L(\pi_n(x_1)) | \leqq | x - x_1 | Dn^2 \left[\delta(b-a) + \sum_{i=1}^{m} | g_i | \right].$$

But $g_i = U_i(\pi_n)$ and hence by (6), $|g_i| \leqq (K/C_i)^{1/r_i} \epsilon_n^s$. Therefore, making use of (7), we can write

$$| g_i | \leqq (K/C_i)^{1/r_i} \frac{\delta}{4(1+m)M},$$

and so $|L(\pi_n(x)) - L(\pi_n(x_1))| \leqq |x - x_1| D'n^2\delta$, where

$$D' = D\left\{ (b-a) + [4(1+m)M]^{-1} \sum_{i=1}^{m} (K/C_i)^{1/r_i} \right\}$$

is a constant independent of n.

We restrict our attention now to a small interval about x_1 in which $|x - x_1| \leqq 1/[2D'n^2]$. In this interval

$$| L(\pi_n(x)) - L(\pi_n(x_1)) | \leqq \delta/2,$$

and since $|L(\pi_n(x_1))| = \delta$, it follows that

$$| L(\pi_n(x)) | \geqq \delta/2.$$

But $|L(F)| \leqq (m+1) M\epsilon_n < (m+1) M\delta/[4(m+1)M] = \delta/4$. Hence in this interval

$$| L(\pi_n) - L(F) | \geqq \delta/4.$$

As the interval from x_1 to $x_1 + 1/(2D'n^2)$ or else that from $x_1 - 1/(2D'n^2)$ to x_1 is contained in (a, b) for n sufficiently large, it is clear that

$$\int_a^b | L(\pi_n - F) |^r dx \geqq \frac{1}{2D'n^2} \left(\frac{\delta}{4} \right)^r.$$

Hence

$$\delta \leqq 4(2D')^{1/r} n^{2/r} \left\{ \int_a^b | L(\pi_n - F) |^r dx \right\}^{1/r},$$

and by virtue of (6) and the fact that $\epsilon_n < \epsilon_n^s$ (ϵ_n being < 1)

$$\delta \leqq 4(2KD')^{1/r} n^{2/r} \epsilon_n^s.$$

This result was proved on the basis of assumption (7), but if (7) does not hold then clearly

$$\delta \leqq 4(1+m)M\epsilon_n^s < 4(1+m)Mn^{2/r}\epsilon_n^s.$$

Hence if E is the larger of the two constants $4(2KD')^{1/r}$ and $4(1+m)M$, we can write in any case

$$\delta \leqq En^{2/r}\epsilon_n^s. \tag{8}$$

We make use of (8) and conclusion (a) of the auxiliary theorem to obtain an upper bound of $|\pi_n^{(k)}(x)|$. Thus for $k=0, 1, \cdots, m$,

$$\begin{aligned}
|\pi_n^{(k)}(x)| &\leqq C\left\{(b-a)\delta + \sum_{i=1}^{m}|g_i|\right\} \\
&\leqq C\left\{(b-a)En^{2/r}\epsilon_n^s + \sum_{i=1}^{m}|U_1(\pi_n)|\right\} \\
&\leqq C\left\{(b-a)En^{2/r}\epsilon_n^s + \left[\sum_{i=1}^{m}(K/C_i)^{1/r_i}\right]\epsilon_n^s\right\} \\
&= C'n^{2/r}\epsilon_n^s,
\end{aligned}$$

where C' is a constant independent of n, the nature of which can be seen from the foregoing expression.

From this last result it follows that

$$\begin{aligned}
|F^{(k)}(x) - \pi_n^{(k)}(x)| &\leqq |F^{(k)}(x)| + |\pi_n^{(k)}(x)| \leqq \epsilon_n + C'n^{2/r}\epsilon \\
&\leqq C''n^{2/r}\epsilon_n^s,
\end{aligned}$$

where $C''=(1+C')$ is a constant independent of n. But, as we have already observed in the first case, $F(x)-\pi_n(x)$ is identical with $y(x)-P_n(x)$ and hence

$$|y^{(k)}(x) - P_n^{(k)}(x)| \leqq C''n^{2/r}\epsilon_n^s$$

for $k=0, 1, \cdots, m$, and for all values of x on (a, b). Thus the approximating process converges if $\lim_{n\to\infty} n^{2/r}\epsilon_n^s = 0$.

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ADDITION FORMULAS FOR HYPERELLIPTIC FUNCTIONS

BY W. V. PARKER

1. *Introduction.** In a recent paper the writer† discussed the Kummer surface associated with the hyperelliptic curve of the form

$$s^2 = r^5 + ar^4 + br^3 + cr^2 + r.$$

This form for the curve was used by A. L. Dixon‡ in a paper in which he obtained addition formulas for hyperelliptic functions with distinct arguments. Dixon emphasizes the unusual symmetry of this particular form for the hyperelliptic curve. In the present paper duplication formulas are found as well as the addition formulas for distinct arguments. The odd functions used here differ from those of Dixon and are slightly different from the ones ordinarily used. The particular form used here seems desirable because of the symmetry.

2. *Distinct Arguments.* Consider the fixed hyperelliptic curve, H, of genus two, given by the equation

$$s^2 = r^5 + ar^4 + br^3 + cr^2 + r, \tag{1}$$

and let L be a variable curve of the form

$$m_0 s = n_0 r^3 + n_1 r^2 + n_2 r + n_3, \qquad m_0 \neq 0. \tag{2}$$

If $n_0 \neq 0$, H and L intersect in six finite points any four of which are sufficient to determine L and hence to determine the remaining two points.

Denote the six points of intersection of H and L by

$$(\alpha_1, \rho_1),\ (\alpha_2, \rho_2),\ (\beta_1, \sigma_1),\ (\beta_2, \sigma_2),\ (\gamma_1, \tau_1),\ (\gamma_2, \tau_2),$$

and let

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* This paper is a part of a dissertation written in Brown University, 1931. The writer is indebted to A. A. Bennett for many helpful suggestions.

† This Bulletin, vol. 38 (1932), p. 403.

‡ *On hyperelliptic functions of genus two*, Quarterly Journal of Mathematics, vol. 36 (1904), p. 1.

$$(3)\quad \begin{aligned} u_1 &= \int_\infty^{\alpha_1} \frac{dr}{s} + \int_\infty^{\alpha_2} \frac{dr}{s}, & u_2 &= \int_\infty^{\alpha_1} \frac{r\,dr}{s} + \int_\infty^{\alpha_2} \frac{r\,dr}{s}, \\ v_1 &= \int_\infty^{\beta_1} \frac{dr}{s} + \int_\infty^{\beta_2} \frac{dr}{s}, & v_2 &= \int_\infty^{\beta_1} \frac{r\,dr}{s} + \int_\infty^{\beta_2} \frac{r\,dr}{s}, \\ w_1 &= \int_\infty^{\gamma_1} \frac{dr}{s} + \int_\infty^{\gamma_2} \frac{dr}{s}, & w_2 &= \int_\infty^{\gamma_1} \frac{r\,dr}{s} + \int_\infty^{\gamma_2} \frac{r\,dr}{s}. \end{aligned}$$

We then have by Abel's Theorem*

$$(4)\quad \begin{aligned} u_1 + v_1 + w_1 &\equiv 0 \pmod{\text{period}}, \\ u_2 + v_2 + w_2 &\equiv 0 \pmod{\text{period}}. \end{aligned}$$

Let

$$\frac{x_\alpha}{t_\alpha} = \alpha_1 + \alpha_2, \qquad \frac{y_\alpha}{t_\alpha} = \alpha_1\alpha_2,$$

$$\eta_\alpha = \frac{\alpha_2\rho_1 - \alpha_1\rho_2}{\alpha_1\alpha_2(\alpha_1 - \alpha_2)}, \qquad \zeta_\alpha = \frac{\alpha_1^2\rho_2 - \alpha_2^2\rho_1}{\alpha_1\alpha_2(\alpha_1 - \alpha_2)};$$

and denote similarly the corresponding expressions in β_1, β_2, σ_1, σ_2; γ_1, γ_2, τ_1, τ_2. These are all periodic functions of the integrals u_1 and u_2. We know that x_α/t_α, and y_α/t_α are even functions† and η_α and ζ_α are odd functions. The functions x_γ/t_γ, y_γ/t_γ, η_γ, ζ_γ can be expressed rationally in terms of x_α, y_α, t_α, η_α, ζ_α, x_β, y_β, t_β, η_β, ζ_β. But x_γ/t_γ, y_γ/t_γ, η_γ, ζ_γ are the same functions of (w_1, w_2) as x_α/t_α, y_α/t_α, η_α, ζ_α are of (u_1, u_2) and x_β/t_β, y_β/t_β, η_β, ζ_β are of (v_1, v_2), and since, from (4),

$$w_1 \equiv -(u_1 + v_1),$$

and

$$w_2 \equiv -(u_2 + v_2),$$

these relations give us expressions for the functions of (u_1+v_1, u_2+v_2) in terms of the functions of (u_1, u_2) and (v_1, v_2), the sign being positive for even functions and negative for odd functions. Furthermore ζ_α can be expressed rationally in terms of x_α, y_α, t_α, η_α, and similarly η_α can be expressed rationally in terms of x_α, y_α, t_α, ζ_α. For the functions are connected by the relations

* Appell et Goursat, *Théorie des Fonctions Algébriques*, Chap. 9.

† Dixon, loc. cit.

$$
\begin{aligned}
y t \eta^2 &= a y t + x y + z t, \\
2 y t \eta \zeta &= b y t - y^2 - t^2 - x z, \\
y t \zeta^2 &= c y t + y z + x t,
\end{aligned} \tag{5}
$$

where

$$
\frac{z}{t} = \frac{2F(\alpha_1, \alpha_2) - 2\rho_1\rho_2}{(\alpha_1 - \alpha_2)^2},
$$

and

$$
\begin{aligned}
2F(\alpha_1, \alpha_2) = {} & \alpha_1^2\alpha_2^2(\alpha_1 + \alpha_2) + 2a\alpha_1^2\alpha_2^2 \\
& + b\alpha_1\alpha_2(\alpha_1 + \alpha_2) + 2c\alpha_1\alpha_2 + \alpha_1 + \alpha_2.
\end{aligned}
$$

We see from these relations that interchanging a with c and y with t also interchanges η with ζ.

If we eliminate s between (1) and (2) we get the equation

$$
\begin{aligned}
& n_0^2 r^6 + (2n_0n_1 - m_0^2)r^5 + (n_1^2 + 2n_0n_2 - am_0^2)r^4 \\
& \quad + (2n_1n_2 + 2n_0n_3 - bm_0^2)r^3 \\
& \quad + (n_2^2 + 2n_1n_3 - cm_0^2)r^2 + (2n_2n_3 - m_0^2)r + n_3^2 = 0,
\end{aligned} \tag{6}
$$

whose roots are $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$. Hence we have

$$
\begin{aligned}
\frac{x_\alpha}{t_\alpha} + \frac{x_\beta}{t_\beta} + \frac{x_\gamma}{t_\gamma} &= \frac{m_0^2 - 2n_0n_1}{n_0^2}, \\
\frac{y_\alpha}{t_\alpha}\,\frac{y_\beta}{t_\beta}\,\frac{y_\gamma}{t_\gamma} &= \frac{n_3^2}{n_0^2}.
\end{aligned} \tag{7}
$$

Since (α_1, ρ_1), (α_2, ρ_2), (β_1, σ_1), (β_2, σ_2) are on L, we have

$$
\begin{aligned}
m_0\rho_1 &= n_0\alpha_1^3 + n_1\alpha_1^2 + n_2\alpha_1 + n_3, \\
m_0\rho_2 &= n_0\alpha_2^3 + n_1\alpha_2^2 + n_2\alpha_2 + n_3, \\
m_0\sigma_1 &= n_0\beta_1^3 + n_1\beta_1^2 + n_2\beta_1 + n_3, \\
m_0\sigma_2 &= n_0\beta_2^3 + n_1\beta_2^2 + n_2\beta_2 + n_3.
\end{aligned} \tag{8}
$$

From (8) we get at once

$$
\begin{aligned}
\eta_\alpha m_0 &= \frac{x_\alpha}{t_\alpha} n_0 + n_1 - \frac{t_\alpha}{y_\alpha} n_3, \quad \zeta_\alpha m_0 = -\frac{y_\alpha}{t_\alpha} n_0 + n_2 + \frac{x_\alpha}{y_\alpha} n_3, \\
\eta_\beta m_0 &= \frac{x_\beta}{t_\beta} n_0 + n_1 - \frac{t_\beta}{y_\beta} n_3, \quad \zeta_\beta m_0 = -\frac{y_\beta}{t_\beta} n_0 + n_2 + \frac{x_\beta}{y_\beta} n_3.
\end{aligned} \tag{9}
$$

If now we let $n_0 = t_\alpha t_\beta p_0$ and $n_3 = y_\alpha y_\beta p_3$ and write

$$\begin{vmatrix} x_\alpha & x_\beta \\ y_\alpha & y_\beta \end{vmatrix} = (x, y), \quad \begin{vmatrix} \eta_\alpha & 1 \\ \eta_\beta & 1 \end{vmatrix} = (\eta, 1), \text{ etc.},$$

we get

$$(10)\quad (\eta, 1)m_0 = (x, t)p_0 + (y, t)p_3, \quad (\zeta, 1)m_0 = (t, y)p_0 + (x, y)p_3.$$

Writing

$$m_0 = \begin{vmatrix} (x, t) & (y, t) \\ (t, y) & (x, y) \end{vmatrix},$$

we have

$$p_0 = \begin{vmatrix} (\eta, 1) & (y, t) \\ (\zeta, 1) & (x, y) \end{vmatrix}, \quad p_3 = \begin{vmatrix} (x, t) & (\eta, 1) \\ (t, y) & (\zeta, 1) \end{vmatrix}.$$

From (9) we get

$$(11)\quad \begin{aligned} 2n_1 &= m_0(\eta_\alpha + \eta_\beta) + p_3(y_\alpha t_\beta + y_\beta t_\alpha) - p_0(x_\alpha t_\beta + x_\beta t_\alpha), \\ 2n_2 &= m_0(\zeta_\alpha + \zeta_\beta) + p_0(y_\alpha t_\beta + y_\beta t_\alpha) - p_3(x_\alpha y_\beta + x_\beta y_\alpha). \end{aligned}$$

It is interesting to note here that interchanging y with t, and η with ζ, interchanges p_0 with p_3, n_0 with n_3, n_1 with n_2, and leaves m_0 unaltered.

If we substitute the value for $2n_1$ from (11) in (7) we get

$$(12)\quad \begin{aligned} \frac{x_\gamma}{t_\gamma} &= \frac{m_0^2 - t_\alpha t_\beta p_0[m_0(\eta_\alpha + \eta_\beta) + p_3(y_\alpha t_\beta + y_\beta t_\alpha)]}{t_\alpha^2 t_\beta^2 p_0^2}, \\ \frac{y_\gamma}{t_\gamma} &= \frac{p_3^2 y_\alpha y_\beta}{p_0^2 t_\alpha t_\beta}. \end{aligned}$$

Since the points (γ_1, τ_1), (γ_2, τ_2) are also on L we have

$$\eta_\gamma m_0 = \frac{x_\gamma}{t_\gamma} n_0 + n_1 - \frac{t_\gamma}{y_\gamma} n_3, \quad \zeta_\gamma m_0 = -\frac{y_\gamma}{t_\gamma} n_0 + n_2 + \frac{x_\gamma}{y_\gamma} n_3.$$

If now we let $x_{\alpha+\beta}$ etc. denote the same functions of (u_1+v_1, u_2+v_2) as x_α etc. are of (u_1, u_2) and x_β etc. are of (v_1, v_2), we have the following addition formulas (valid for distinct arguments):

$$x_\gamma : y_\gamma : t_\gamma = x_{\alpha+\beta} : y_{\alpha+\beta} : t_{\alpha+\beta} = \left\{ \left| \begin{matrix} (x,t) & (y,t) \\ (t,y) & (x,y) \end{matrix} \right|^2 \right.$$

$$- t_\alpha t_\beta \left| \begin{matrix} (\eta,1) & (y,t) \\ (\zeta,1) & (x,y) \end{matrix} \right| \left[(\eta_\alpha + \eta_\beta) \left| \begin{matrix} (x,t) & (y,t) \\ (t,y) & (x,y) \end{matrix} \right| \right.$$

(13)

$$\left. + (y_\alpha t_\beta + y_\beta t_\alpha) \left| \begin{matrix} (x,t) & (\eta,1) \\ (t,y) & (\zeta,1) \end{matrix} \right| \right] \left. \right\} : y_\alpha y_\beta t_\alpha t_\beta \left| \begin{matrix} (x,t) & (\eta,1) \\ (t,y) & (\zeta,1) \end{matrix} \right|^2$$

$$: t_\alpha^2 t_\beta^2 \left| \begin{matrix} (\eta,1) & (y,t) \\ (\zeta,1) & (x,y) \end{matrix} \right|^2,$$

$$\eta_\gamma = -\eta_{\alpha+\beta} = \frac{m_0}{t_\alpha t_\beta p_0} - \frac{p_0(x_\alpha t_\beta + x_\beta t_\alpha)}{2m_0} - \frac{p_0^2 t_\alpha t_\beta}{m_0 p_3}$$

$$- \frac{p_3(y_\alpha t_\beta + y_\beta t_\alpha)}{2m_0} - \frac{\eta_\alpha + \eta_\beta}{2},$$

$$\zeta_\gamma = -\zeta_{\alpha+\beta} = \frac{m_0}{y_\alpha y_\beta p_3} - \frac{p_3(x_\alpha y_\beta + x_\beta y_\alpha)}{2m_0} - \frac{p_3^2 y_\alpha y_\beta}{m_0 p_0}$$

$$- \frac{p_0(y_\alpha t_\beta + y_\beta t_\alpha)}{2m_0} - \frac{\zeta_\alpha + \zeta_\beta}{2}.$$

These last two equations may be written

$$\eta_\gamma = -\eta_{\alpha+\beta} = \frac{m_0}{t_\alpha t_\beta p_0} + \frac{p_0 t_\alpha t_\beta (\zeta,1)}{p_3(y,t)} - \frac{\eta_\alpha y_\alpha t_\beta - \eta_\beta y_\beta t_\alpha}{(y,t)},$$

$$\zeta_\gamma = -\zeta_{\alpha+\beta} = \frac{m_0}{y_\alpha y_\beta p_3} + \frac{p_3 y_\alpha y_\beta (\eta,1)}{p_0(t,y)} - \frac{\zeta_\alpha y_\beta t_\alpha - \zeta_\beta y_\alpha t_\beta}{(t,y)}.$$

From (12) we get

$$\frac{x_\gamma}{y_\gamma} = \frac{m_0^2 - t_\alpha t_\beta p_0 [m_0(\eta_\alpha + \eta_\beta) + p_3(y_\alpha t_\beta + y_\beta t_\alpha)]}{y_\alpha y_\beta t_\alpha t_\beta p_3^2}.$$

If now we interchange y with t, η with ζ and p_0 with p_3 we get

$$\frac{x_\gamma}{t_\gamma} = \frac{m_0^2 - y_\alpha y_\beta p_3 [m_0(\zeta_\alpha + \zeta_\beta) + p_0(y_\alpha t_\beta + y_\beta t_\alpha)]}{y_\alpha y_\beta t_\alpha t_\beta p_0^2}.$$

This is exactly the form obtained for x_γ/t_γ by using the coefficient of r from (6) rather than the coefficient of r^5. A somewhat

more symmetric though much longer form for x_γ/t_γ is obtained by taking one-half the sum of these two expressions.

3. *The Coincidence Case, Duplication Formulas.* If (β_1, σ_1) coincides with (α_1, ρ_1) and also (β_2, σ_2) coincides with (α_2, ρ_2), that is, if $u_1 = v_1$ and $u_2 = v_2$, the above formulas become indeterminate in the sense that they do not give the expressions for the functions of $(2u_1, 2u_2)$ in terms of the functions of (u_1, u_2) simply by setting $v_1 = u_1$ and $v_2 = u_2$. In order to obtain the formulas in this case as expeditiously as possible we determine the curve L so that it is tangent to H at each of the points (α_1, ρ_1) and (α_2, ρ_2). We then have by Abel's theorem

$$2u_1 + w_1 \equiv 0 \text{ (mod period)}, \quad 2u_2 + w_2 \equiv 0 \text{ (mod period)}.$$

Since the roots of (6) are now $\alpha_1, \alpha_1, \alpha_2, \alpha_2, \gamma_1, \gamma_2$, we have

$$\frac{x_\gamma}{t_\gamma} = \frac{m_0^2 - 2n_0n_1}{n_0^2} - \frac{2x_\alpha}{t_\alpha}, \qquad \frac{y_\gamma}{t_\gamma} = \frac{n_3^2 t_\alpha^2}{n_0^2 y_\alpha^2}, \tag{14}$$

where the ratios n_0/m_0, n_1/m_0, n_2/m_0, n_3/m_0 are to be determined from the equations*

$$yt\eta m_0 = xyn_0 + ytn_1 - t^2n_3, \quad yt\zeta m_0 = -y^2n_0 + ytn_2 + xtn_3, \tag{15}$$

$$(5xy^2 + 4ay^2t - 2cyt^2 - xt^2)m_0 = 6(y^2t\zeta + xy^2\eta)n_0 + 4y^2t\eta n_1 - 2yt^2\zeta n_2,$$

$$(5y^3 - 3by^2t - 2cxyt - x^2t + yt^2)m_0 = 6y^3\eta n_0 - 4y^2t\zeta n_1 - 2(xyt\zeta + y^2t\eta)n_2.$$

The computation here is greatly simplified by introducing the function z/t as defined in (5) which is expressed rationally in terms of x/t, y/t, η, ζ through the relations given there. Making use of these relations and writing

$$\begin{aligned} p_0 &= (y^2 - t^2)\eta + (xy - zt)\zeta, & p_2 &= t\eta^2 + z\eta\zeta + y\zeta^2, \\ p_1 &= y\eta^2 + x\eta\zeta + t\zeta^2, & p_3 &= (xt - yz)\eta + (t^2 - y^2)\zeta, \end{aligned} \tag{16}$$

we get immediately

$$\begin{aligned} m_0 &= 2ytp_1, \quad n_0 = tp_0, \quad n_1 = tp_3 - xp_0 + 2ytp_1\eta, \\ n_2 &= yp_0 - xp_3 + 2ytp_1\zeta, \quad n_3 = yp_3. \end{aligned} \tag{17}$$

* Since there is no possibility of confusion the subscript α will be omitted in what is to follow.

If now $x_{2\alpha}$ etc. denote the same functions of $(2u_1, 2u_2)$ as x_α etc. are of (u_1, u_2), and we substitute the expressions from (17) in (14), we get the following duplication formulas:

$$x_\gamma : y_\gamma : t_\gamma = x_{2\alpha} : y_{2\alpha} : t_{2\alpha} = (4yt p_1 p_2 - 2p_0 p_3) : p_3^2 : p_0^2 .$$

Since the points (γ_1, τ_1) and (γ_2, τ_2) are on L, we have

$$\eta_\gamma m_0 = \frac{x_\gamma}{t_\gamma} n_0 + n_1 - \frac{t_\gamma}{y_\gamma} n_3, \quad \zeta_\gamma m_0 = -\frac{y_\gamma}{t_\gamma} n_0 + n_2 + \frac{x_\gamma}{y_\gamma} n_3.$$

Hence we have

$$\eta_\gamma = -\eta_{2\alpha} = \frac{[(4yt p_1 p_2 - 2p_0 p_3)t - x p_0^2]p_3 + p_0(t p_3^2 - y p_0^2)}{2yt p_0 p_1 p_3} + \eta,$$

$$\zeta_\gamma = -\zeta_{2\alpha} = \frac{[(4yt p_1 p_2 - 2p_0 p_3)y - x p_3^2]p_0 + p_3(y p_0^2 - t p_3^2)}{2yt p_0 p_1 p_3} + \zeta,$$

where p_0, p_1, p_2, p_3 are as defined in (16).

For certain choices of (α_1, ρ_1) and (α_2, ρ_2) in the coincidence case, (γ_1, τ_1) will coincide with (γ_2, τ_2) and the curve L will be tangent to H at each of three points. In fact (α_1, ρ_1) can be chosen arbitrarily subject to the condition that $\rho_1 \neq 0$ and then (α_2, ρ_2) can be determined in a finite number of ways so that the curve L which is tangent to H at (α_1, ρ_1) and (α_2, ρ_2) will also be tangent at a third point. If we take a fixed point on H and determine a curve L through this point and any three of the points where H meets the r-axis, this curve L will meet H in another pair of points such that the L which is tangent to H at each of these is also tangent at the given fixed point. Furthermore all such tangent curves may be obtained in this way.*

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* For proof of this statement see a paper by the writer in this Bulletin, vol. 37 (1931), p. 557.

A CORRECTION IN THE LIST OF PRIMES

BY D. N. LEHMER

My attention has been called by Franz Merkout of Gorlitz, Germany, to an error in carrying the hundreds digit from the bottom of one column to the top of the next in the *List of Primes* published by the Carnegie Institution of Washington in 1914. On page 11, at the top of the 13th column, the number in the table should be 628051 instead of 628151. The latter number is not a prime. All users of these tables should insert this correction in their tables. It is the first one so far discovered.

Professor Kempner, of the University of Colorado, has pointed out that errors in determining the rank of a given prime would be less likely if the columns had been numbered from 0 to 49 on the pages instead of from 1 to 50. This is perhaps true. At any rate the author in the introduction has made the mistake of listing the prime 5,815,627 as the 401,108th prime instead of the 401,008th. In spite of this palpable testimony as to the chance of a slip with the present arrangement of the table I am inclined to think the safety after making such a change would not be appreciably increased.

FORTY-FIRST ANNUAL LIST OF PAPERS

READ BEFORE THE AMERICAN MATHEMATICAL SOCIETY AND SUBSEQUENTLY PUBLISHED, INCLUDING REFERENCES TO THE PLACES OF PUBLICATION

ADAMS, C. R. On summability of double series. Read Sept. 8, 1931. *Transactions of this Society*, vol. 34, No. 2, pp. 215–230; April, 1932.

—— Note on multiple Dirichlet and multiple factorial series. Read Sept. 9, 1931. *Annals of Mathematics*, (2), vol. 33, No. 3, pp. 406–412; July, 1932.

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—— On equivalence of methods of evaluation of sequences. Read April 3, 1931. *Tôhoku Mathematical Journal*, vol. 35, No. 2, pp. 244–252; May, 1932.

—— On deferred Cesàro means. Read Sept. 10, 1931. *Annals of Mathematics*, (2), vol. 33, No. 3, pp. 413–421; July, 1932.

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—— Normal division algebras of degree 2^e and pure Riemann matrices. Read Dec. 28, 1931. *Annals of Mathematics*, (2), vol. 33, No. 2, pp. 311–318; April, 1932.

—— Normal division algebras of degree four over an algebraic field. Read Oct. 31, 1931. *Transactions of this Society*, vol. 34, No. 2, pp. 363–372; April, 1932.

—— A construction of non-cyclic normal division algebras. Read April 9, 1932. *This Bulletin*, vol. 38, No. 6, pp. 449–456; June, 1932.

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ERRATA

Page 406, line 25: *for* $-2xy^3z$ *read* $-2xy^2z$.

Page 482, line 27: *for* Plaschke *read* Blaschke.

Page 490, line 9: *for* Mr. H. G. Russell *read* Dr H. G. Russell.

Page 546, line 22: *for* $(\gamma u+\delta v-\epsilon u)$ *read* $(\gamma w+\delta v-\epsilon u)$.

Page 629, line 36: *for* T. C. Bumer *read* C. T. Bumer.

OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

NEW YORK CITY, February 25, 1933.

Abstracts must be in the hands of the Secretary of the Society, 501 West 116th St., New York City, not later than February 4, 1933. Abstracts received by January 7 will appear in the January issue of the BULLETIN. By invitation of the program committee, Professor F. R. Sharpe will deliver an address entitled *The algebraic theory of involutorial transformation.*

STANFORD UNIVERSITY, March 18, 1933.

Abstracts must be in the hands of Associate Secretary T. M. Putnam, University of California, Berkeley, California, not later than February 25. Abstracts received by that date will appear in the March issue of the BULLETIN.

NEW YORK CITY, April 14-15, 1933.

Abstracts must be in the Office of the Society, 501 West 116th St., New York City, not later than March 24. Abstracts received by March 6 will appear in the March issue of the BULLETIN. A symposium on algebra is being arranged. The leading speakers will be Professor Oystein Ore, on *Abstract ideal theory and applications*, and Professor J. V. Uspensky, *On the theory of numbers and integral calculus.*

CHICAGO, April 14-15, 1933.

Abstracts must be in the hands of the Associate Secretary, Mark H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than March 25. Abstracts received by March 6 will appear in the March issue of the BULLETIN. Professor C. C. MacDuffee will deliver a symposium lecture on *Matrices with elements in a principal ideal ring.*

R. G. D. RICHARDSON, *Secretary of the Society.*

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

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The initiation fees and the annual dues of members of the Society (see this BULLETIN, p. 322, May, 1930; and the List of Officers and Members, September, 1932, p. 66), are payable to the Treasurer of the Society, Professor G. W. Mullins, 501 West 116th St., New York City.

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